Fluctuations of linear eigenvalue statistics of β matrix models in the multi-cut regime

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Abstract

We study the asymptotic expansion in n for the partition function of β matrix models with real analytic potentials in the multi-cut regime up to the $O(n^{-1})$ terms. As a result, we find the limit of the generating functional of linear eigenvalue statistics and the expressions for the expectation and the variance of linear eigenvalue statistics, which in the general case contain the quasi periodic in n terms.

1 Introduction and main results

In this paper we consider a class of distributions in $\mathbb{R}^n$ of the form

$$p_{n,\beta}(\lambda_1,...,\lambda_n)=Q_{n,\beta}^{-1}[V]\prod_{i=1}^{n}e^{-n\beta V(\lambda_i)/2}\prod_{1\le i<j\le n}|\lambda_i-\lambda_j|^\beta=Q_{n,\beta}^{-1}e^{\beta H(\lambda_1,...,\lambda_n)/2},\tag{1.1}$$

where the function H, which we call Hamiltonian to stress the analogy with statistical mechanics, and the normalizing constant $Q_{n,\beta}[V]$ (called the partition function) have the form

$$H(\lambda_1,\dots,\lambda_n)=-n\sum_{i=1}^{n}V(\lambda_i)+\sum_{i\ne j}\log|\lambda_i-\lambda_j|,\tag{1.2}$$

$$Q_{n,\beta}[V]=\int e^{\beta H(\lambda_1,...,\lambda_n)/2}d\lambda_1\dots d\lambda_n.\tag{1.3}$$

We denote also

$$\mathbf{E}_{n,\beta}\{(\dots)\}=\int(\dots)p_{n,\beta}(\lambda_1,...,\lambda_n)d\lambda_1,\dots d\lambda_n,\tag{1.4}$$

$$p_{n,\beta}^{(l)}(\lambda_1,...,\lambda_l)=\int_{\mathbb{R}^{n-l}}p_{n,\beta}(\lambda_1,...\lambda_l,\lambda_{l+1},...,\lambda_n)d\lambda_{l+1}...d\lambda_n\tag{1.5}$$

the corresponding expectation and the lth marginal densities (correlation functions) of (1.1). The function V in (1.2), called the potential, is a real valued Hölder function satisfying the condition

$$V(\lambda)\ge 2(1+\epsilon)\log(1+|\lambda|).\tag{1.6}$$

Such distributions can be considered for any $\beta>0$, but the cases $\beta=1,2,4$ are especially important, since they correspond to real symmetric, hermitian, and symplectic matrix models respectively.

Since the papers [3, 9] it is known that if V is a Hölder function, then

$$n^{-2}\log Q_{n,\beta}[V]=\frac{\beta}{2}\mathcal{E}[V]+O(\log n/n),$$

where

$$\mathcal{E}[V]=\max_{m\in\mathcal{M}_1}\left\{L[dm,dm]-\int V(\lambda)m(d\lambda)\right\}=\mathcal{E}_V(m^*),\tag{1.7}$$

and the maximizing measure m^* (called the equilibrium measure) has a compact support $\sigma:=\operatorname{supp}m^*$. Here and below we denote

$$\begin{aligned}L[\,dm,dm]&=\int\log|\lambda-\mu|dm(\lambda)dm(\mu),\\ L[f](\lambda)&=\int\log|\lambda-\mu|f(\mu)d\mu,\quad L[f,g]=(L[f],g),\end{aligned}\tag{1.8}$$

where $(.,.)$ is a standard inner product in $L_2[\mathbb{R}]$.

If V' is a Hölder function, then the equilibrium measure m^* has a density ρ (equilibrium density). The support σ and the density ρ are uniquely defined by the conditions:

$$\begin{array}{ll}v(\lambda):=2\displaystyle\int\log|\mu-\lambda|\rho(\mu)d\mu-V(\lambda)=\sup v(\lambda):=v^*,\quad\lambda\in\sigma,\\ v(\lambda)\le\sup v(\lambda),\quad\lambda\notin\sigma,\qquad\qquad\sigma=\operatorname{supp}\{\rho\}.\end{array}\tag{1.9}$$

Without loss of generality we will assume below that $\sigma\subset(-1,1)$ and $v^*=0$.

In this paper we discuss the asymptotic expansion in n^{-k} of the partition function $Q_{n,\beta}[V]$ and of the Stieltjes transforms of the marginal densities. The problems of this kind appear in many fields of mathematics, e.g., statistical mechanics of log-gases, combinatorics (graphical enumeration), theory of orthogonal polynomials etc (see [5] for the detailed and interesting discussion on the motivation of the problem). Here we are going to discuss with more details the applications of the problem to the analysis of the eigenvalue distributions of random matrices.

One of the most important problems of the eigenvalue distribution is the behavior of the random variables, called the linear eigenvalue statistics, corresponding to the smooth test function h

$$\mathcal{N}_n[h]=\sum_{i=1}^n h(\lambda_i).\tag{1.10}$$

The result of [3] gives us the main term of the expectation of $\mathbf{E}_{n,\beta}\{\mathcal{N}_n[h]\}$ which is $n(h,\rho)$. It was also proven in [3] that the variance of $\mathcal{N}_n[h]$ tends to zero, as $n\to\infty$. But the behavior of the fluctuations of $\mathcal{N}_n[h]$ was studied only in the case of one-cut potentials (see [9]). Even the bound for $\mathbf{Var}_{n,\beta}\{\mathcal{N}_n[h]\}$ in the multi-cut regime till the recent time was known only for $\beta=2$. Thus the behavior of the characteristic functional, corresponding to the linear eigenvalue statistics (1.10) of the test function h

$$Z_{n,\beta}[h]=\mathbf{E}_{n,\beta}\Big\{e^{\beta(\mathcal{N}_n[h]-\mathbf{E}_{n,\beta}\{\mathcal{N}_n[h]\})/2}\Big\}=\frac{Q_{n,\beta}\big[V-\frac{1}{n}(h-\mathbf{E}_{n,\beta}\{n^{-1}\mathcal{N}_n[h]\})\big]}{Q_{n,\beta}\big[V\big]}.\tag{1.11}$$

is one of the questions of primary interest in the random matrix theory. Since $Z_{n,\beta}[h]$ is a ratio of two partition functions, to study the behavior of $Z_{n,\beta}[h]$, it suffices to find the coefficients of the expansion of $\log Q_{n,\beta}\big[V\big]$ up to the order $O(n^{-1})$.

Let us mention the most important results on the expansion of $\log Q_{n,\beta}[V]$ and the correlation functions. The CLT for linear eigenvalue statistics in the one-cut regime for any β and polynomial V was proven in [9]. The expansion for the first and the second correlators for $\beta = 2$ and one-cut real analytic V was constructed in [1]. The expansion of $\log Q_{n,\beta}[V]$ for a one-cut polynomial V and $\beta = 2$ was obtained in [5]. The formal expansions for any β and polynomial V were obtained in the physical papers [4] and [7]. The CLT for $\beta = 2$, real analytic multi-cut V, and special choice $h = V$ was obtained in [12]. The expansion of $\log Q_{n,\beta}[V]$ up to $O(1)$ for one-cut real analytic V and multi-cut real analytic V was performed in [11] and [16] respectively. The complete asymptotic expansion of the partition function and all the correlators for one-cut real analytic V and any β was constructed in [2]. It worth to mention that the papers [9, 11, 2] are based on the same method, the first version of which was proposed in [9]. The method is based on the analysis of the first loop equation by the methods of the perturbation theory, where the results of [3] give zero order approximation. The subsequent papers [11, 2] simplified and developed the method of [9]. This allowed to the authors to extend the method to non-polynomial V (see [11]), and to apply it to the loop equations of higher orders (see [2]). As a result in [2] the complete asymptotic expansion of the partition function and all the correlators were constructed. The essential disadvantage of this method is that it is not applicable to the multi-cut case. A method which allows to factorize $Q_{n,\beta}[V]$ in the multi cut case to the product of the partition functions of the one cut "effective" potentials, was proposed in [16]. In the present paper the same idea is used to study the limit of the characteristic functional $Z_{n,\beta}[h]$ and to construct the expansion of $Q_{n,\beta}[V]$ up to $o(1)$ terms (see Theorem 2). We assume the following conditions:

Condition C1. *V is a real analytic potential satisfying (1.6). The support of the equilibrium measure is*

$$\sigma = \bigcup_{\alpha=1}^{q} \sigma_\alpha, \quad \sigma_\alpha = [a_\alpha, b_\alpha]; \tag{1.12}$$

Condition C2. *The equilibrium density ρ can be represented in the form*

$$\rho(\lambda) = \frac{1}{2\pi} P(\lambda) \Im X_\sigma^{1/2}(\lambda + i0), \quad \inf_{\lambda\in\sigma} |P(\lambda)| > 0, \tag{1.13}$$

where

$$X_\sigma(z) = \prod_{\alpha=1}^{q} (z - a_\alpha)(z - b_\alpha), \tag{1.14}$$

and we choose a branch of $X_\sigma^{1/2}(z)$ such that $X_\sigma^{1/2}(z) \sim z^q$, as $z \to +\infty$. Moreover, the function v defined by (1.9) attains its maximum only if λ belongs to σ.

Remark 1 *It is known (see, e.g., [1]) that for analytic V the equilibrium density ρ always has the form (1.13) – (1.14). The function P in (1.13) is analytic and can be represented in the form*

$$P(z) = \frac{1}{2\pi i} \oint_{\mathcal{L}} \frac{V'(z) - V'(\zeta)}{(z - \zeta) X_\sigma^{1/2}(\zeta)} d\zeta. \tag{1.15}$$

Hence condition C2 means that ρ has no zeros in the internal points of σ and behaves like square root near the edge points. This behavior of V is usually called generic.

The first result of the paper is the theorem which allows us to control $Z_{n,\beta}[h]$ and $\log Q_{n,\beta}[V]$ in the one cut case up to $O(n^{-1})$ terms. The essential difference with similar results of [9], [11] and [2] is that Theorem 1 is applicable to a non real h. This fact is very important because the proof of Theorem 2 is based on the application of Theorem 1 to a non real h. Besides, since the results of [2] were obtained for real analytic h, the remainder bounds found here cannot be used in the proof of Theorem 2.

Theorem 1 *Let V satisfy (1.6), the equilibrium density ρ (see (1.9)) have the form (1.13) with $q=1$, and $\sigma=\operatorname{supp}\rho=[a,b]$. Assume also that V is analytic in the domain $\mathbf{D}\supset\sigma_\varepsilon$, where σ_ε is the ε-neighborhood of σ. Then:*
(i) For any real h with $||h^{(6)}||_\infty, ||h'||_\infty\le n^{1/2}\log n$ the characteristic functional $Z_{n,\beta}[h]$ has the form

$$Z_{n,\beta}[h]=\exp\Big\{\frac{\beta}{2}\Big((\frac{2}{\beta}-1)(h,\nu)+\frac{1}{4}(\overline{D}h,h)\Big)+n^{-1}O\big(||h'||_\infty^3+||h^{(6)}||_\infty^3\big)\Big\},\qquad(1.16)$$

where the operator $\overline{D}_\sigma$ is defined by

$$\overline{D}_\sigma=\frac{1}{2}(D_\sigma+D_\sigma^*),\quad D_\sigma h(\lambda)=\frac{X^{-1/2}(\lambda)}{\pi^2}\int_\sigma\frac{h'(\mu)X^{1/2}(\mu)d\mu}{(\lambda-\mu)},\qquad(1.17)$$

and a non positive measure ν has the form

$$(\nu,h):=\frac{1}{4}(h(b)+h(a))-\frac{1}{2\pi}\int_\sigma\frac{h(\lambda)d\lambda}{X^{1/2}(\lambda)}-\frac{1}{2}(D_\sigma\log P,h)\qquad(1.18)$$

with P defined by (1.15) and $X^{1/2}(\lambda):=\Im X^{1/2}(\lambda+i0)$ with X of (1.14). Here and below $||h||_\infty=\sup_{\lambda\in\sigma_\varepsilon}|h(\lambda)|$.
(ii) If h is non real and $|\beta(D_\sigma h,h)|\le\kappa\log n$ with some absolute κ and $||h^{(6)}||_\infty\le n^{1/6}$, then

$$Z_{n,\beta}[h]=\exp\Big\{\frac{\beta}{2}\Big((\frac{2}{\beta}-1)(h,\nu)+\frac{1}{4}(\overline{D}_\sigma h,h)\Big)\}\Big\}\Big(1+n^{-1/2}O\big(||h'||_\infty^3+||h^{(6)}||_\infty^3\big)\Big).\quad(1.19)$$

(iii) For $h=0$

$$\begin{aligned}\log(Q_{n,\beta}/n!)=&\frac{\beta n^2}{2}\mathcal{E}[V]+F_\beta(n)+n\Big(\frac{\beta}{2}-1\Big)\big((\log\rho,\rho)-1-\log 2\pi\big)\\&+r_\beta[\rho]+O(n^{-1}),\end{aligned}\qquad(1.20)$$

where $r_\beta[\rho]$ is given by the integral representation of (2.25) and $F_\beta(n)$ corresponds to the linear, logarithmic and zero order terms of the expansion in n of $\log Q_{n,\beta}[V^*]$ for $V^*(\lambda)=\lambda^2/2$. According to [8]

$$F_\beta(n)=n\Big(\frac{\beta}{2}-1\Big)\Big(\log\frac{n\beta}{2}-\frac{1}{2}\Big)+n\log\frac{\sqrt{2\pi}}{\Gamma(\beta/2)}-c_\beta\log n+c_\beta^{(1)},\qquad(1.21)$$

where $c_\beta=\frac{\beta}{24}-\frac{1}{4}+\frac{1}{6\beta}$ and $c_\beta^{(1)}$ is some constant/ depending only on β (for $\beta=2$, $c_\beta^{(1)}=\zeta'(1)$).

Remark 2 *Let us note that the operator D_σ is "almost" $(-\mathcal{L}_\sigma)^{-1}$, where $\mathcal{L}_\sigma$ is the integral operator defined by (1.8) for the interval σ. More precisely, if we denote $X_\sigma^{-1/2}=\mathbf{1}_\sigma|X_\sigma^{-1/2}|$ with X_σ of (1.14)*

$$\begin{aligned}D_\sigma\mathcal{L}_\sigma v=-v+\pi^{-1}(v,\mathbf{1}_\sigma)X_\sigma^{-1/2},\quad&\mathcal{L}_\sigma D_\sigma v=-v+\pi^{-1}(v,X_\sigma^{-1/2})\mathbf{1}_\sigma,\\\mathcal{L}_\sigma D_\sigma^* v=-v+\pi^{-1}(v,X_\sigma^{-1/2})\mathbf{1}_\sigma,\quad\Rightarrow\quad&\mathcal{L}_\sigma\bar{D}_\sigma v=-v+\pi^{-1}(v,X_\sigma^{-1/2})\mathbf{1}_\sigma.\end{aligned}\qquad(1.22)$$

Remark 3 *For $\beta = 2$ in the one-cut case we have*

$$\log(Q_{n,2}/n!) = n^2\mathcal{E}[V] + \frac{n}{2}\log 2\pi - \frac{1}{12}\log n + \zeta'(1) - \frac{2}{3(b-a)^2}\log\frac{P(a)P(b)}{P_0^2} + O(n^{-1}), \quad (1.23)$$

where $P_0 = 16/(b-a)^2$ corresponds to the Gaussian potential $V_0(\lambda) = 2(2\lambda - a - b)^2/(b-a)^2$, such that the support of its equilibrium measure is $[a,b]$.

Consider the space

$$\mathcal{H} = \oplus_{\alpha=1}^{q} L_1[\sigma_\alpha]. \quad (1.24)$$

Note that we need $\mathcal{H}$ mainly as a set of functions, its topology is not important below. Define the operator $\mathcal{L}$ as

$$\mathcal{L}f = \mathbf{1}_\sigma L[f\mathbf{1}_\sigma], \quad \widehat{\mathcal{L}}_\alpha f := \mathbf{1}_{\sigma_\alpha} L[f\mathbf{1}_{\sigma_\alpha}]. \quad (1.25)$$

Moreover, we will consider the block diagonal operators

$$\overline{D} := \oplus_{\alpha=1}^{q}\overline{D}_\alpha, \quad \widehat{\mathcal{L}} := \oplus_{\alpha=1}^{q}\widehat{\mathcal{L}}_\alpha, \quad (1.26)$$

where $\overline{D}_\alpha$ is defined by (1.17) for σ_α. Introduce also

$$\widetilde{\mathcal{L}} := \mathcal{L} - \widehat{\mathcal{L}}, \quad \mathcal{G} := (1 - \overline{D}\widetilde{\mathcal{L}})^{-1}. \quad (1.27)$$

An important role below belongs to a positive definite matrix of the form

$$\mathcal{Q} = \{\mathcal{Q}_{\alpha\alpha'}\}_{\alpha,\alpha'=1}^{q}, \quad \mathcal{Q}_{\alpha\alpha'} = -(\mathcal{L}\psi^{(\alpha)}, \psi^{(\alpha')}), \quad (1.28)$$

where $\psi^{(\alpha)}(\lambda) = p_\alpha(\lambda)X^{-1}(\lambda)\mathbf{1}_\sigma$ (p_α is a polynomial of degree $q-1$) is a unique solution of the system of equations

$$-(\mathcal{L}\psi^{(\alpha)})_{\alpha'} = \delta_{\alpha\alpha'}, \quad \alpha' = 1, \dots, q. \quad (1.29)$$

Denote also

$$I[h] = (I_1[h], \dots, I_q[h]), \quad I_\alpha[h] := \sum_{\alpha'} \mathcal{Q}^{-1}_{\alpha\alpha'}(h, \psi^{(\alpha')}). \quad (1.30)$$

$$\mu_\alpha = \int_{\sigma_\alpha}\rho_\alpha(\lambda)d\lambda, \quad \rho_\alpha := \mathbf{1}_{\sigma_\alpha}\rho. \quad (1.31)$$

The main result of the paper is the following theorem:

Theorem 2 *Let the potential V satisfy conditions C1-C2, and let $||h^{(6)}||_\infty < \infty$. Then*

$$Z_{n,\beta}[h] = e^{\frac{\beta}{8}(\mathcal{G}\overline{D}h,h) + \left(\frac{\beta}{2}-1\right)(\mathcal{G}\nu,h)}\frac{\Theta(\bar{I}[h];\{n\bar{\mu}\})}{\Theta(0;\{n\bar{\mu}\})}\big(1 + O\big(n^{-\kappa}(||h'||_\infty^3 + ||h^{(6)}||_\infty^3)\big)\big), \quad (1.32)$$

where

$$\Theta(I[h];\{n\bar{\mu}\}) := \sum_{n_1+\dots+n_q=n_0} \exp\Big\{ -\frac{\beta}{2}\Big(\mathcal{Q}^{-1}\Delta\bar{n}, \Delta\bar{n}\Big) + \frac{\beta}{2}(\Delta\bar{n}, I[h]) + \big(\frac{\beta}{2}-1\big)(\Delta\bar{n}, I[\log\overline{\rho}])\Big\}, \quad (1.33)$$

$$\{n\bar{\mu}\} = (\{n\mu_1\}, \dots, \{n\mu_q\}), \quad (\Delta\bar{n})_\alpha = n_\alpha - \{n\mu_\alpha\}, \quad n_0 = \sum_{\alpha=1}^{q}\{n\mu_\alpha\},$$

with a positive definite matrix $\mathcal{Q}$ of (1.28), $I[h]$ defined by (1.30), and $\log\overline{\rho} = (\log\rho_1, \dots, \log\rho_q)$.

For $h = 0$ we have

$$
\begin{aligned}
Q_{n,\beta}[V] =& \mathcal{Z}_{n,\beta} \frac{\exp\Big\{\dfrac{2}{\beta}\Big(\dfrac{\beta}{2}-1\Big)^2(\widetilde{\mathcal{L}}\mathcal{G}\nu,\nu)\Big\}}{\det^{1/2}(1-\overline{D}\widetilde{\mathcal{L}})}\Theta(0;\{n\bar{\mu}\})(1+O(n^{-\kappa})), \qquad (1.34)\\
\mathcal{Z}_{n,\beta}[V] =& \exp\Big\{\frac{n^2\beta}{2}\mathcal{E}[V] + F_\beta(n) + n(\frac{\beta}{2}-1)\big((\log\rho,\rho)-1-\log 2\pi\big)\\
&- c_\beta(q-1)\log n + \sum_{\alpha=1}^{q}(r_\beta[\mu_\alpha^{-1}\rho_\alpha] - c_\beta\log\mu_\alpha)\Big\},
\end{aligned}
$$

where μ_α, ρ_α are defined in (1.31), $r_\beta[\rho]$ is defined in (2.25), $F_\beta(n)$ and c_β are defined in (1.21) and det *means the Fredholm determinant of $\overline{D}\widetilde{\mathcal{L}}$ on σ.*

It is evident that Theorem 1 yields that the fluctuations of $\mathcal{N}_n[h]$ for generic h are non Gaussian. They are Gaussian, if there exists some c such that

$$
I_\alpha[h] = c, \quad \alpha = 1, \dots, q; \quad \Leftrightarrow \quad (h-c, \psi^{(\alpha)}) = 0, \quad \alpha = 1, \dots, q. \qquad (1.35)
$$

Moreover, inspecting the proof of Theorem 2, one can see that it is proven in fact that $\log Z_{n,\beta}[th]$ is an analytic function of t for some small enough t. Since

$$
n(p_{n,\beta}^{(1)} - \rho, h) = \frac{2}{\beta}\partial_t \log Z_{n,\beta}[th]\Big|_{t=0}, \quad \mathbf{Var}_{n,\beta}\{\mathcal{N}_n[h]\} = \big(\frac{2}{\beta}\big)^2\partial_t^2\log Z_{n,\beta}[th]\Big|_{t=0},
$$

one can find $n(p_{n,\beta}^{(1)} - \rho, h)$ and $\mathbf{Var}_{n,\beta}\{\mathcal{N}_n[h]\}$, differentiating the r.h.s. of (1.32). It is easy to see that if conditions (1.35) are not fulfilled, then both expressions contain the derivatives of $\log\Theta(I(h);\{n\mu\})$, hence they are quasi periodic functions.

Let us note that relations (1.22) imply

$$
-\mathcal{L}\mathcal{G}\overline{D} = (1 + P^{(1)}\widetilde{\mathcal{L}}\mathcal{L}^{-1})^{-1}(1-P^{(1)}) = 1 + P^{(1)}\widehat{F},
$$

where $P^{(1)}$ is a block-diagonal operator $P_\alpha^{(1)}v = (v, X_\alpha^{-1/2})\mathbf{1}_{\sigma_\alpha}$ and $\widehat{F}$ is some operator. Hence

$$
(\mathcal{G}\overline{D}h)(\lambda) = -(\mathcal{L}^{-1}h)(\lambda) + \sum c_\alpha(h)\psi^{(\alpha)}(\lambda),
$$

where $c_\alpha(v)$ are some constants and $\psi^{(\alpha)}$ are defined by (1.29). Besides, evidently $\mathcal{G}\overline{D}\mathbf{1}_{\sigma_\alpha} = 0$, and therefore

$$
0 = (\mathcal{G}\overline{D}h, \mathbf{1}_{\sigma_\alpha}) = -(\mathcal{L}^{-1}h, \mathbf{1}_{\sigma_\alpha}) + \sum_{\alpha'} Q_{\alpha\alpha'}c_\alpha(h), \quad \alpha = 1, \dots, q.
$$

These conditions determine $c_\alpha(h)$ uniquely. On the other hand, if we define the operator $\mathcal{D}_\sigma$ by the formula (1.17) with X_σ from (1.14) for the multi cut case, then it has the same form with some $\widetilde{c}_\alpha(h)$. Hence

$$
\mathcal{G}\overline{D} = \mathcal{D}_\sigma + \sum_\alpha \psi^{(\alpha)} \otimes f^{(\alpha)},
$$

where $f^{(\alpha)}$ are some functions of the form $X_\sigma^{-1/2}p_\alpha$ with some polynomials p_α.

The paper is organized as follows. The proofs of Theorem 1 and Theorem 2 are given in Section 2 and Section 3 respectively. Proofs of some auxiliary results, used in the proof of Theorem 2, are given in Section 4.

2 Proof of Theorem 1

To prove Theorem 1 we study the Stieltjes transform

$$g_{n,\beta,h}(z) = \int \frac{p^{(1)}_{n,\beta,h}(\lambda)d\lambda}{z-\lambda} \tag{2.1}$$

of the first marginal density $p^{(1)}_{n,\beta,h}$ defined by (1.5) for V replaced by $V - \frac{1}{n}h$. Let us represent

$$g_{n,\beta,h} = g + n^{-1}u_{n,\beta,h},$$

where g is the Stieltjes transform of the equilibrium density ρ. According to [16],

$$u_{n,\beta,h}(z) = (\mathcal{K}F)(z), \tag{2.2}$$

where the operator $\mathcal{K} : \mathrm{Hol}[\mathbf{D} \setminus \sigma_\varepsilon] \to \mathrm{Hol}[\mathbf{D} \setminus \sigma_\varepsilon]$ is defined by the formula

$$(\mathcal{K}f)(z) := \frac{1}{2\pi i X^{1/2}(z)} \oint_{\mathcal{L}} \frac{f(\zeta)d\zeta}{P(\zeta)(z-\zeta)}, \tag{2.3}$$

with the contour $\mathcal{L}$ which does not contain z and zeros of P, and

$$\begin{aligned} F(z) = &\int \frac{h'(\lambda)p^{(1)}_{n,\beta,h}(\lambda)}{z-\lambda}d\lambda - \Big(\frac{2}{\beta}-1\Big)g'(z) \\ &- \frac{2/\beta-1}{n}u'_{n,\beta,h}(z) + \frac{1}{n}u^2_{n,\beta,\eta}(z) + \frac{1}{n}\delta_{n,\beta,h}(z), \end{aligned} \tag{2.4}$$

with

$$\delta_{n,\beta,h}(z) = \int \frac{n(n-1)p^{(2)}_{n,\beta,h}(\lambda,\mu) - n^2 p^{(1)}_{n,\beta,h}(\lambda)p^{(1)}_{n,\beta,h}(\mu) + n\delta(\lambda-\mu)p^{(1)}_{n,\beta,h}(\lambda)}{(z-\lambda)(z-\mu)}d\lambda d\mu. \tag{2.5}$$

Moreover, according to [16] $u_{n,\beta,h}$ and $\delta_{n,\beta,h}$ satisfy the bounds:

$$|u_{n,\beta,h}(z)| \le C_0\frac{\log n}{d^{5/2}(z)}(1+||h'||_\infty), \quad |\delta_{n,\beta,h}(z)| \le C(1+||h'||_\infty)^2\frac{\log^2 n}{d^5(z)}, \tag{2.6}$$

if $d(z) := \mathrm{dist}\{z, \sigma_\varepsilon\} \ge n^{-1/3}\log n$. In addition,

$$|(p_{n,\beta,h} - \rho, \varphi)| \le Cn^{-1}(||\varphi'''||_\infty + ||\varphi'||_\infty). \tag{2.7}$$

Using (2.6) in (2.2) we get for $d(z) > n^{-1/3}\log n$

$$\begin{aligned} u_{n,\beta,h}(z) =&(\mathcal{K}\widehat{h})(z) - \Big(\frac{2}{\beta}-1\Big)(\mathcal{K}g')(z) \\ &+ n^{-1}\Big((1+||h'||^2_\infty)O(d^{-11/2}(z)) + ||h^{(4)}||_\infty O(d^{-3/2}(z))\Big), \end{aligned} \tag{2.8}$$

where

$$\widehat{h}(z) := \int \frac{h'(\lambda)\rho(\lambda)}{z-\lambda}d\lambda.$$

We note here that although (2.8) was obtained for z inside the domain $\mathbf{D}_2$ where V is an analytic function and which does not contain zeros of P, we can extend (2.8) to $z \notin \mathbf{D}_2$, using that $u_{n,\beta,h}(z)$ is analytic everywhere in $\mathbb{C} \setminus \sigma_\varepsilon$ and behaves like $|u_{n,\beta,h}(z)| \sim nz^{-2}$, as $z \to \infty$. Applying the Cauchy theorem, we have for any $z \notin \mathbf{D}_2$

$$u_{n,\beta,h}(z) = \frac{1}{2\pi i} \oint_L \frac{u_{n,\beta,h}(\zeta) d\zeta}{z - \zeta}$$

with the contour $L \subset \mathbf{D}_2$.

Let us transform

$$\begin{aligned}(\mathcal{K}\widehat{h})(z) =& \frac{1}{(2\pi i)^2 X^{1/2}(z)} \oint_{\mathcal{L}} \frac{d\zeta}{(z-\zeta)P(\zeta)} \int \frac{h'(\lambda)P(\lambda)|X^{1/2}(\lambda)|}{\zeta - \lambda} d\lambda \\ =& \frac{X^{-1/2}(z)}{2\pi} \int \frac{h'(\lambda)|X^{1/2}(\lambda)|}{z - \lambda} d\lambda.\end{aligned} \tag{2.9}$$

Similarly, we have

$$\begin{aligned}-(\mathcal{K}g')(z) =& \frac{1}{2\pi i X^{1/2}(z)} \oint_{\mathcal{L}} \int \frac{\rho(\lambda) d\zeta d\lambda}{(\zeta - \lambda)^2 P(\zeta)(z - \zeta)} \\ =& -\frac{X^{-1/2}(z)}{2\pi} \int_{\mathcal{L}} \frac{(\log P(\lambda))' X^{1/2}(\lambda) d\lambda}{(z - \lambda)} - \frac{1}{2} \frac{z - c}{X(z)} + \frac{1}{2X^{1/2}(z)}.\end{aligned} \tag{2.10}$$

Hence, we obtain that for $\varphi_z(\lambda) = (z - \lambda)^{-1}$ with $d(z) \geq n^{-1/3} \log n$

$$\begin{aligned}n(p_{n,\beta,h} - \rho, \varphi_z) =& (\frac{2}{\beta} - 1)(\nu, \varphi_z) + \frac{1}{2}(Dh, \varphi_z) \\ &+ n^{-1}\Big((1 + ||h'||_\infty^2) O(d^{-11/2}(z)) + ||h^{(4)}||_\infty O(d^{-3/2}(z))\Big),\end{aligned} \tag{2.11}$$

where ν is defined by (1.18).

To extend (2.11) on the differentiable φ, consider the Poisson kernel

$$\mathcal{P}_y(\lambda) = \frac{y}{\pi(y^2 + \lambda^2)}.$$

It is easy to see that for any integrable φ

$$(\mathcal{P}_y * \varphi)(\lambda) = \frac{1}{\pi} \Im \int \frac{\varphi(\mu) d\mu}{\mu - (\lambda + iy)}.$$

Hence (2.11) implies

$$\begin{aligned}&||\mathcal{P}_y * \nu_{n,\beta,h}||_2^2 \leq Cn^{-1}\Big(1 + ||h'||_\infty^4) y^{-11} + ||h^{(4)}||_\infty^2 y^{-3}\Big), \quad |y| \geq n^{-1/3} \log n, \\ &\nu_{n,\beta,h}(\lambda) := n\Big(p_{\beta,h}^{(n)}(\lambda) - \rho(\lambda)\Big) - (\frac{2}{\beta} - 1)\nu(\lambda) - \frac{1}{2} Dh(\lambda),\end{aligned} \tag{2.12}$$

where $||.||_2$ is the standard norm in $L_2(\mathbb{R})$ and the sign measure ν is defined in (1.18).

Then we use the following formula (see [9]) valid for any sign measure ν

$$\int_0^\infty e^{-y} y^{2s-1} ||\mathcal{P}_y * \nu_{n,\beta,h}||_2^2 dy = \Gamma(2s) \int_{\mathbb{R}} (1 + 2|\xi|)^{-2s} |\widehat{\nu}_{n,\beta,h}(\xi)|^2 d\xi. \tag{2.13}$$

This formula for $s = 6$, the Parseval equation for the Fourier integral, and the Schwarz inequality yield

$$\begin{aligned}
\int_{\mathbb{R}} \varphi(\lambda)\nu_{n,\beta,h}(\lambda)d\lambda =& \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{\varphi}(\xi)\widehat{\nu}_{n,\beta,h}(\xi)d\xi \\
\leq& \frac{1}{2\pi}\left(\int_{\mathbb{R}} |\widehat{\varphi}(\xi)|^2(1+2|\xi|)^{2s}d\xi\right)^{1/2}\left(\int_{\mathbb{R}} |\widehat{\nu}_{n,\beta,h}(\xi)|^2(1+2|\xi|)^{-2s}d\xi\right)^{1/2} \\
\leq& \frac{C((||\varphi||_2+||\varphi^{(6)}||_2)}{\Gamma^{1/2}(2s)}\left(\int_0^\infty e^{-y}y^{2s-1}||P_y * \nu_{n,\beta,h}||_2^2 dy\right)^{1/2} \\
\leq& Cn^{-1}((||\varphi||_2+||\varphi^{(6)}||_2)(1+||h'||_\infty^2+||h^{(4)}||_\infty).
\end{aligned}$$

To estimate the last integral here, we split it into two parts $|y| \geq n^{-1/3}\log n$ and $|y| < n^{-1/3}\log n$. For the first integral we use (2.12) and for the second - the bound (see [14])

$$|u_{n,h}(z)| \leq \frac{C^* n^{1/2}\log^{1/2} n}{d(z)},$$

where C^* is an n, η-independent constant which depends on $||V' + \frac{1}{n}h'||_\infty$, ε, and $|b-a|$. Thus we get that for any function φ with bounded sixth derivative

$$\begin{aligned}
n(p^{(1)}_{n,\beta,h} - \rho, \varphi) =& (\frac{2}{\beta} - 1)(\nu, \varphi_z) + (Dh, \varphi_z) \\
&+ (||\varphi||_2 + ||\varphi^{(6)}||_2)(1+||h'||_\infty^2 + ||h^{(4)}||_\infty)O(n^{-1}).
\end{aligned} \tag{2.14}$$

Since

$$\frac{d}{dt}\log Z_{n,\beta}[th] = \int_{\sigma_\varepsilon} h(\lambda)p^{(1)}_{n,\beta,th}(\lambda),$$

integrating (2.14) with $\varphi = th$ with respect to t, we get (1.16) for real h. To extend this relation to the complex valued h we use the following lemma.

Lemma 1 *Let $\{X_n\}_{n\geq 1}$ be a sequence of random variables such that*

$$\mathbf{E}\{e^{tX_n}\} = e^{t^2/2}(1 + O(n^{-1}\log^{3/2} n)), \quad -\log^{1/2} n \leq t \leq \log^{1/2} n. \tag{2.15}$$

Then the relation

$$\mathbf{E}\{e^{tX_n}\} = e^{t^2/2}(1 + O(n^{-1/2}\log^{3/2} n)), \tag{2.16}$$

holds in the circle $\frac{1}{7}D$, where $D = \{t : |t| \leq \log^{1/2} n\}$

Proof. Consider a strip $S = \{t : |\Re t| \leq \log^{1/2} n\}$. It is evident that $\mathbf{E}\{e^{tX_n}\}$ is analytic in S and bounded by $2\sqrt{n}$ for sufficiently big n. Introduce the analytic function

$$f_n(t) := c(e^{-t^2/2}\mathbf{E}\{e^{tX_n}\} - 1)n/\log^{3/2} n, \quad t \in D,$$

where we choose the constant $c > 0$ such that

$$|f_n(t)| \leq 1, \quad t \in \gamma = [-\log^{1/2} n, \log^{1/2} n].$$

It is possible by (2.15). Moreover, $f_n(t) \le n^2$, $t \in D$. Then, by the theorem on two constants (see [6]), we conclude that

$$\log|f_n(t)| \le 2(1-\omega(t;\gamma,D'))\log n,$$

where $\omega(t;\gamma,D')$ is the harmonic measure of the set γ with respect to the domain D' at the point $t \in D'$, where $D' := D \cap \mathbb{C}_+$. It is well-known (see again [6]) that

$$\omega(t;\gamma,D') = 1 - \frac{2}{\pi}\Im\log\frac{1+t\log^{-1/2}n}{1-t\log^{-1/2}n}.$$

Hence $1-\omega(t;\gamma,D') \le 14\Im t/(3\pi\log^{1/2}n)$ for $t \in \frac{1}{7}D'$, and we obtain from the above inequalities that

$$\log|f_n(t)| \le \frac{28\log^{1/2}n}{3\pi}\Im t, \quad t \in \frac{1}{7}D'.$$

We finally deduce from the last bound that

$$\log|f_n(t)| \le \frac{1}{2}\log n, \quad \Rightarrow \quad |f_n(t)| \le n^{1/2}, \ t \in \frac{1}{7}D',$$

and from the definition of f_n we obtain (2.16).

□

(iii) To prove (1.20) we need to control $u_{n,\beta,h}$ up to the order n^{-1}. It follows from (2.2) and (2.4) that for this aim we need to control zero order term of $u_{n,\beta,h}$ (which is known already) and zero order term of $\delta_{n,\beta,h}(z)$. It is easy to see that if we replace $h(\lambda)$ by $h_t^{(\alpha)} = h(\lambda) + th_{z_0}(\lambda)$ with $h_{z_0}(\lambda) = (\lambda - z_0)^{-1}$, then

$$\delta_{n,\beta,h}(z_0) = \partial_t u_{n,\beta,h_t}(z_0)\Big|_{t=0}.$$

It was proven in [9] that $u_{n,\beta,h_t}(z)$ is an analytic function of t for small enough t. Hence, integrating with respect to t over the circle $|t| = C_0 d^2(z_0)/2$, we get that for any $||h'|| \le C_0/2$

$$\partial_t u_{n,\beta,h_t}(z)\Big|_{t=0} = \frac{1}{\pi X_\eta^{1/2}(z)}\oint_{\mathcal{L}_d}\frac{h'_{z_0}(\lambda)|X^{1/2}(\lambda)|d\lambda}{(z-\lambda)} + n^{-1}O(d^{-11/2}(z)d^{-2}(z_0)).$$

Thus we obtain for $h = 0$ and any real analytic V, satisfying conditions C1-C2:

$$\delta_{n,\beta}(z) = \frac{1}{\pi X^{1/2}(z)}\int_\sigma \frac{X^{1/2}(\lambda)}{(\lambda-z)^3}d\lambda + n^{-1}O(d^{-15/2}(z)) = \frac{1}{X^2(z)} + n^{-1}O(d^{-15/2}(z)). \quad (2.17)$$

Set

$$\begin{aligned} &V^{(0)}(z) = 2(z-c)^2/d^2, \quad c = (a+b)/2, \quad d = (b-a)/2, \quad P_0 = 4/d^2, \\ &g_t(z) = tg(z) + \frac{2(1-t)}{d^2}(z - c - X^{1/2}(z)), \quad P_t(\lambda) = P_0 + t(P(\lambda) - P_0), \end{aligned}$$

and consider the functions V_t of the form

$$V_t(\lambda) = V^{(0)}(\lambda) + t\Delta V(\lambda), \quad \Delta V(\lambda) = V(\lambda) - V^{(0)}(\lambda). \quad (2.18)$$

Let $Q_{n,\beta}(t) := Q_{n,\beta}[V_t]$ be defined by (1.28) with V replaced by V_t. Then, evidently, $Q_{n,\beta}(1) = Q_{n,\beta}[V]$, and $Q_{n,\beta}(0)$ corresponds to $V^{(0)}$. Hence

$$\begin{aligned} \frac{1}{n^2}\log Q_{n,\beta}(1) - \frac{1}{n^2}\log Q_{n,\beta}(0) &= \frac{1}{n^2}\int_0^1 dt\frac{d}{dt}\log Q_{n,\beta}(t) \\ &= -\frac{\beta}{2}\int_0^1 dt\int d\lambda\Delta V(\lambda)p_{n,\beta}^{(1)}(\lambda;t), \end{aligned} \tag{2.19}$$

where $p_{n,\beta}^{(1)}(\lambda;t)$ is the first marginal density corresponding to V_t. Using (1.9), one can check that for the distribution (1.1) with V replaced by V_t the equilibrium density ρ_t has the form

$$\rho_t(\lambda) = t\rho(\lambda) + (1-t)\rho^{(0)}(\lambda), \quad \rho^{(0)}(\lambda) = \frac{2X^{1/2}(\lambda)}{\pi d^2}, \quad \Delta\rho(\lambda) = \rho(\lambda) - \rho_0(\lambda) \tag{2.20}$$

with X, d of (1.20). Using (2.2), (2.4), (2.8), and (2.17), one can write:

$$g_n(z,t) = g(z,t) + n^{-1}u_\beta^{(0)}(z,t) + n^{-2}u_\beta^{(1)}(z,t) + O(n^{-3}), \tag{2.21}$$

where

$$\begin{aligned} u_\beta^{(0)}(z,t) &= -\Big(\frac{2}{\beta} - 1)(\mathcal{K}_t g_t')(z), \\ u_\beta^{(1)}(z,t) &= \mathcal{K}_t\Big((u_\beta^{(0)})^2 - (2/\beta - 1)\partial_z u_\beta^{(0)} + \frac{1}{X^2}\Big)(z,t), \end{aligned} \tag{2.22}$$

and the operator $\mathcal{K}_t$ is defined by (2.3) with P replaced by $P_t = P_0 + (1-t)P$.

Substituting (2.21) in the last integral in (2.19), we get

$$\begin{aligned} \log Q_{n,\beta}[V] =& \log Q_{n,\beta}[V^{(0)}] - n^2\frac{\beta}{2}\mathcal{E}[V^{(0)}] + n^2\frac{\beta}{2}\mathcal{E}[V] \\ &+ n(\frac{\beta}{2} - 1)\int_0^1 dt(\Delta V(\lambda), \nu_t) - \int_0^1 dt\frac{\beta}{4\pi i}\oint \Delta V(z)u_\beta^{(1)}(z,t)dz + O(n^{-1}). \end{aligned} \tag{2.23}$$

Write $\Delta V = 2L[\Delta\rho] + v^{(0)}$ where $v^{(0)}$ is a constant from (1.29), corresponding to $V^{(0)}$ (recall that we assumed that corresponding constant for V, is zero). Then, taking into account (1.18), we get

$$(\Delta V(\lambda), \nu) = \frac{1}{4}(\Delta V(a) + \Delta V(b)) - \frac{v^{(0)}}{2} - (L[\Delta\rho], D\log P_t).$$

Then (1.22) yields

$$(L[\Delta\rho], D\log P_t) = (\Delta\rho, LD\log P_t) = -(\Delta\rho, \log P_t).$$

Now we can integrate with respect to t and obtain

$$\begin{aligned} \int_0^1 dt(\Delta V(\lambda), \nu) &= \frac{1}{4}(\Delta V(a) + \Delta V(b)) - \frac{v^{(0)}}{2} \\ &+ \int_\sigma \rho(\lambda)\log P(\lambda)d\lambda - \int_\sigma \rho_0(\lambda)\log P_0(\lambda)d\lambda \\ &= \int_\sigma \rho(\lambda)\log\rho(\lambda)d\lambda - 1 - \log 2\pi + \log(d/2), \end{aligned} \tag{2.24}$$

since

$$\int_\sigma \rho(\lambda)\log X^{1/2}(\lambda)d\lambda - \frac{1}{4}(V(a)+V(b)) = \frac{1}{2}(L[\rho](a)+L[\rho](b)) - \frac{1}{4}(V(a)+V(b)) = 0,$$
$$\int_\sigma \rho_0(\lambda)\log P_0 d\lambda = -2\log(d/2) \quad V^{(0)}(a) = V^{(0)}(b) = 2, \quad v^{(0)} = 2\log(d/2).$$

In addition, changing the variables in the corresponding integrals, we have

$$\log Q_{n,\beta}[V^{(0)}] = \log Q^*_{n,\beta} + \Big(\frac{n^2\beta}{2} + n(1-\beta/2)\Big)\log\frac{d}{2},$$
$$\frac{n^2\beta}{2}\mathcal{E}[V^{(0)}] = -\frac{3n^2\beta}{8} + \frac{n^2\beta}{2}\log\frac{d}{2}.$$

These relations combined with (2.23), (2.24) and (2.22) imply (1.20) with

$$\begin{aligned} r_\beta[\rho] :=& -\frac{1}{2\pi i}\int_0^1 dt\oint_L \Delta V(z)u_\beta^{(1)}(z,t)dz \\ =& \frac{1}{(2\pi)^2}\int_0^1 dt\oint_L dz\frac{\Delta V(z)}{X^{1/2}(z)}\oint_{L'} d\zeta\frac{\Big((u^{(0)})^2 - (\frac{2}{\beta}-1)\partial_\zeta u^{(0)} + X^{-2}\Big)(\zeta,t)}{(z-\zeta)(P_0+t\Delta P(\zeta))}, \end{aligned} \tag{2.25}$$

where the contour L contains L', which contains σ_ε, all zeros of P_t are outside of L, and $u_\beta^{(0)}$ is defined in (2.22). For $\beta = 2$ $u_\beta^{(0)} = 0$, hence we can leave only $X^{-2}(\zeta)$ in the last numerator and take the integral with respect to ζ. Taking into account that

$$\Delta V'(z) = 2\Delta g(z) + \Delta P(z)X^{1/2}(z),$$

and $\Delta g(z) \sim Cz^{-2}$, as $z\to\infty$, we have

$$\begin{aligned} \frac{a-b}{2}\oint_L \frac{\Delta V(z)dz}{X^{1/2}(z)(z-a)} &= \oint_L \frac{\Delta V'(z)(z-b)^{1/2}dz}{(z-a)^{1/2}} \\ &= \oint_L \frac{2\Delta g(z)(z-b)^{1/2}dz}{(z-a)^{1/2}} + \oint_L \frac{\Delta P(z)X^{1/2}(z)(z-b)^{1/2}dz}{(z-a)^{1/2}} = 0, \\ \oint_L \frac{\Delta V(z)dz}{X^{1/2}(z)(z-a)^2} &= \frac{2}{3}\oint_L \frac{\Delta V'(z)dz}{(z-b)^{1/2}(z-a)^{3/2}} - \frac{1}{3}\oint_L \frac{\Delta V(z)dz}{(z-b)^{3/2}(z-a)^{3/2}} \\ &= \frac{2}{3}\oint_L \frac{\Delta P(z)dz}{(z-a)} = \frac{4\pi i}{3}\Delta P(a), \end{aligned}$$

and similar relation for integrals with $(z-a)$ replaced by $(z-b)$. Thus we obtain (1.23). □

3 Proof of Theorem 2

Denote

$$\begin{aligned} &\sigma_\varepsilon = \bigcup_{\alpha=1}^q \sigma_{\alpha,\varepsilon}, \quad \sigma_{\alpha,\varepsilon} = [a_\alpha - \varepsilon, b_\alpha + \varepsilon], \\ &\operatorname{dist}\{\sigma_{\alpha,\varepsilon}, \sigma_{\alpha',\varepsilon}\} > \delta > 0, \quad \alpha \neq \alpha'. \end{aligned} \tag{3.1}$$

It is known (see [3, Lemmas 1,3] and [14, Theorems 11.1.4, 11.1.6]) that if we replace in (1.1) and (1.5) the integration over $\mathbb{R}$ by the integration over σ_ε, then the new partition function $Q^{(\varepsilon)}_{n,\beta}[V]$ and the old one $Q_{n,\beta}[V]$ satisfy the inequality

$$\big|Q_{n,\beta}[V]/Q^{(\varepsilon)}_{n,\beta}[V]-1\big|\le e^{-n\beta d_\varepsilon}$$

Thus, it suffices to study $Q^{(\varepsilon)}_{n,\beta}[V]$ instead of $Q_{n,\beta}[V]$. Starting from this moment, we assume that the replacement of the integration domain is made, but we will omit superindex ε.

Consider the "approximating" function H_a (Hamiltonian)

$$H_a(\lambda_1\dots\lambda_n)=-\,n\sum V^{(a)}(\lambda_i)+\sum_{i\ne j}\log|\lambda_i-\lambda_j|\Big(\sum_{\alpha=1}^{q}\mathbf{1}_{\sigma_{\alpha,\varepsilon}}(\lambda_i)\mathbf{1}_{\sigma_{\alpha,\varepsilon}}(\lambda_j)\Big)-n^2\Sigma^*, \quad (3.2)$$

$$V^{(a)}(\lambda)=\sum_{\alpha=1}^{q}V^{(a)}_\alpha(\lambda),\quad V^{(a)}_\alpha(\lambda)=\mathbf{1}_{\sigma_{\alpha,\varepsilon}}(\lambda)\Big(V(\lambda)-2\int_{\sigma\setminus\sigma_\alpha}\log|\lambda-\mu|\rho(\mu)d\mu\Big), \quad (3.3)$$

where $V^{(a)}_\alpha(\lambda)$ is the "effective potential". It is easy to check that (1.9) implies

$$V^{(a)}_\alpha=2L[\rho_\alpha]. \quad (3.4)$$

The "cross energy" Σ^* in (3.2) has the form

$$\Sigma^*:=\sum_{\alpha\ne\alpha'}L[\rho_\alpha,\rho_{\alpha'}]. \quad (3.5)$$

Then

$$\begin{aligned}
H(\lambda_1\dots\lambda_n)&=H_a(\lambda_1\dots\lambda_n)+\Delta H(\lambda_1\dots\lambda_n),\quad \lambda_1,\dots,\lambda_n\in\sigma_\varepsilon, \quad (3.6)\\
\Delta H(\lambda_1\dots\lambda_n)&=\sum_{i\ne j}\log|\lambda_i-\lambda_j|\sum_{\alpha\ne\alpha'}\mathbf{1}_{\sigma_{\alpha,\varepsilon}}(\lambda_i)\mathbf{1}_{\sigma_{\alpha',\varepsilon}}(\lambda_j)-2n\sum_{j=1}^{n}\widetilde{V}(\lambda_j)+n^2\Sigma^*,\\
\widetilde{V}(\lambda)&=\sum_{\alpha=1}^{q}\mathbf{1}_{\sigma_{\alpha,\varepsilon}}(\lambda)\int_{\sigma\setminus\sigma_\alpha}\log|\lambda-\mu|\rho(\mu)d\mu.
\end{aligned}$$

Set

$$\overline{n}:=(n_1,\dots,n_q),\quad |\bar n|:=\sum_{\alpha=1}^{q}n_\alpha,\quad \mathbf{1}_{\overline{n}}(\overline{\lambda}):=\prod_{j=1}^{n_1}\mathbf{1}_{\sigma_{1,\varepsilon}}(\lambda_j)\cdots\prod_{j=|\bar n|-n_q+1}^{n}\mathbf{1}_{\sigma_{q,\varepsilon}}(\lambda_j). \quad (3.7)$$

The key observation which explains our motivation to introduce H_a and ΔH is that

$$\begin{aligned}
\mathbf{1}_{\overline{n}}(\overline{\lambda})\Delta H(\overline{\lambda})&=\mathbf{1}_{\overline{n}}(\overline{\lambda})\sum_{\alpha\ne\alpha'}\sum_{j,k=1}^{n}\mathbf{1}_{\sigma_{\alpha,\varepsilon}}(\lambda_j)\mathbf{1}_{\sigma_{\alpha',\varepsilon}}(\lambda_k) \quad (3.8)\\
&\cdot\int\log|\lambda-\mu|(\delta(\lambda_j-\lambda)-\frac{n}{n_\alpha}\rho(\lambda))(\delta(\lambda_k-\mu)-\frac{n}{n_{\alpha'}}\rho(\mu))d\lambda d\mu.
\end{aligned}$$

It was proven in [16] that $\mathbf{E}_{n,\beta}\{\Delta H\}=O(1)$, hence this term is "smaller" than H_a. On the other hand, by the construction, H_a does not contain an "interaction" between different

intervals σ_α, so it is possible to apply to it the result of Theorem 1. This idea was used in [16] to prove that $Q_{\bar{n},\beta}[V]$ can be factorized into a product of one-cut partition functions corresponding to $V_\alpha^{(a)}$ with the error $O(1)$. Here we are doing the next step.

It is easy to see that if we denote

$$Q_{\bar{n},\beta}[V] = \int_{\sigma_\varepsilon^n} \mathbf{1}_{\overline{n}}(\overline{\lambda}) e^{\beta H(\lambda_1 \ldots \lambda_n)/2} d\lambda_1 \ldots d\lambda_n, \tag{3.9}$$

then

$$\frac{Q_{n,\beta}[V]}{n!} = \sum_{|\bar{n}|=n} \frac{Q_{\bar{n},\beta}[V]}{n_1! \ldots n_q!}. \tag{3.10}$$

According to the result of [16],

$$\frac{Q_{\bar{n},\beta}[V]}{n_1! \ldots n_q!} \Big/ \frac{Q_{n,\beta}[V]}{n!} \le e^{-c(\Delta n, \Delta n)}, \quad \Delta n := (n_1 - \mu_1 n, \ldots, n_q - \mu_q n),$$

where μ_α were defined in (1.31). Hence, to construct the expansion of $Q_{n,\beta}[V]$, it is enough to consider in (3.10) only those terms for which

$$(\Delta n, \Delta n) \le \log^2 n. \tag{3.11}$$

To manage with these terms we are going "to linearize" the quadratic form (3.8) by using the integral Gaussian representation (see (3.16) below). Then we will apply Theorem 1 inside the integrals and then integrate the result. As the first step in this direction one should find a good approximation of the integral quadratic form (3.8) by some quadratic form of the finite rank. To this aim consider the space of functions

$$\mathcal{H}_\varepsilon = \oplus_{\alpha=1}^q L_1[\sigma_{\alpha,2\varepsilon}],$$

and the operator $\mathcal{L}$ with the kernel $\log|\lambda - \mu|$. It has a block structure $\{\mathcal{L}_{\alpha,\alpha'}\}_{\alpha,\alpha'=1}^q$. Denote $\widehat{\mathcal{L}}$ its block-diagonal part and by $\widetilde{\mathcal{L}}$ the off diagonal part.

Consider the Chebyshev polynomials $\{p_k^{(\alpha)}\}_{k=0}^\infty$ on $\sigma_{\alpha,2\varepsilon}$ the corresponding orthonormal system of the functions

$$p_k^{(\alpha)}(\lambda) = \cos k \Big(\arccos \Big(\frac{2\lambda - (a_\alpha + b_\alpha)}{b_\alpha - a_\alpha + 4\varepsilon} \Big) \Big), \quad \varphi_k^{(\alpha)}(\lambda) = p_k^{(\alpha)}(\lambda) |X_{\sigma_{\alpha,2\varepsilon}}^{-1/4}(\lambda)|.$$

It is well known that $\{\varphi_k^{(\alpha)}\}_{k=0}^\infty$ make an orthonormal basis in $L_2[\sigma_{\alpha,2\varepsilon}]$, hence we can write

$$\mathbf{1}_{\sigma_{\alpha,\varepsilon}}(\lambda) \mathbf{1}_{\sigma_{\alpha',\varepsilon}}(\mu) \log|\lambda - \mu| = \sum_{k_\alpha, k_{\alpha'}=1}^\infty \mathcal{L}_{k,\alpha;k',\alpha'} p_k^{(\alpha)}(\lambda) p_{k'}^{(\alpha')}(\mu),$$

$$\mathcal{L}_{k,\alpha;k',\alpha'} = \int\int \log|\lambda - \mu| \frac{p_k^{(\alpha)}(\lambda)}{|X_{\sigma_{\alpha,2\varepsilon}}^{1/2}(\lambda)|} \frac{p_{k'}^{(\alpha')}(\mu)}{|X_{\sigma_{\alpha',2\varepsilon}}^{1/2}(\mu)|} d\lambda d\mu.$$

Proposition 1 *There exists $C, d > 0$ such that for all $\alpha \ne \alpha'$*

$$|\mathcal{L}_{k,\alpha;k',\alpha'}| \le C e^{-d(k+k')}. \tag{3.12}$$

The proof of the proposition is given in Section 4.

Proposition 1 implies, in particular, that if we choose $M = [\log^2 n]$, then uniformly in λ, μ

$$\begin{aligned}\mathbf{1}_{\sigma_{\alpha,\varepsilon}}(\lambda)\mathbf{1}_{\sigma_{\alpha',\varepsilon}}(\mu)\log|\lambda-\mu| =& \mathbf{1}_{\sigma_{\alpha,\varepsilon}}(\lambda)\mathbf{1}_{\sigma_{\alpha',\varepsilon}}(\mu)\sum_{k,k'=1}^{M}\mathcal{L}_{k,\alpha;k',\alpha'}p_k^{(\alpha)}(\lambda)p_{k'}^{(\alpha')}(\mu) \\ &+ O(e^{-d\log^2 n}).\end{aligned} \tag{3.13}$$

Consider the matrix $\widetilde{\mathcal{L}}^{(M)} := \{\mathcal{L}_{k,\alpha;k',\alpha'}\}_{\substack{k,k'=1,\dots,M;\\ \alpha,\alpha'=1,\dots,q,\alpha\neq\alpha'}}$. It is a symmetric block matrix in which the block $\{\mathcal{L}_{k,\alpha;k',\alpha'}\}_{k,k'=1,\dots,M}$ corresponds to the kernel $\widetilde{\mathcal{L}}^{(M)}_{\alpha\alpha'}$ which is the r.h.s. sum of (3.13).

Now we would like to represent the matrix $\widetilde{\mathcal{L}}^{(M)}$ as a difference of two positive matrices. To this aim consider the integral operator $\mathcal{A}$ in $\mathcal{H}_\varepsilon$ with a kernel $a(|\lambda-\mu|)$ of the form

$$a(\lambda) = \begin{cases} \log d^{-1} + a_0(\lambda/d) - a_0(1), & 0 \le \lambda \le d, \\ \log|\lambda|^{-1}, & d \le \lambda, \end{cases} \tag{3.14}$$

where the function

$$a_0(\lambda) = \frac{3}{4}\lambda^4 - \frac{8}{3}\lambda^3 + 3\lambda^2$$

is chosen in such a way that $a(\lambda)$ and its first 4 derivatives are continuous at $\lambda = d$, and the third derivative of $a(|\lambda|)$ has a jump at $\lambda = 0$.

Lemma 2 *The integral operator A with the kernel $a(|\lambda-\mu|)$ is positive in $L_2(\Delta)$ where $\Delta \subset [-1,1]$ is any finite system of intervals in $\mathbb{R}$. Moreover, the integral operator with a kernel* $\log|\lambda-\mu|^{-1} - a(|\lambda-\mu|)$ *is positive in $L_2(\Delta)$*

Remark 4 *One can easily see that if we choose $a_0(\lambda) = \lambda - 1$, then the operator A will be also positive, but in this case the Fourier transform $\widehat{a}(k) \sim k^{-2}$, as $k \to \infty$, while we need below $\widehat{a}(k) \sim k^{-4}$.*

Let $\widehat{\mathcal{A}}$ be a block-diagonal part of $\mathcal{A}$. By the construction and the lemma we have

$$\widetilde{\mathcal{L}} = \widehat{\mathcal{A}} - \mathcal{A}, \quad \mathcal{A} \ge 0, \quad \widehat{\mathcal{A}} \ge 0, \quad \widehat{\mathcal{A}} \le \widehat{\mathcal{L}} \tag{3.15}$$

By (3.15), if we consider the matrix of $\mathcal{A}^{(M)}$ and $\widehat{\mathcal{A}}^{(M)}$ at the same basis we obtain

$$\mathcal{L}_{k,\alpha;k',\alpha'} = \mathcal{A}^{(M)}_{k,\alpha;k',\alpha'} - \widehat{\mathcal{A}}^{(M)}_{k,\alpha;k',\alpha'}.$$

Since $\mathcal{A}^{(M)}$ and $\widehat{\mathcal{A}}^{(M)}$ are positive matrices they can be written in the form $\mathcal{A}^{(M)} = S^2$, $\widehat{\mathcal{A}}^{(M)} = \widehat{S}^2$. Thus

$$\begin{aligned}\Delta H =& \sum_{j,\alpha'}\Big(\sum_{l=1}^{n}\sum_{k,\alpha}(\widehat{S}_{j,\alpha';k,\alpha}\big(p_k^{(\alpha)}(\lambda_l) - c_k^{(\alpha)}\big)\Big)^2 \\ &- \sum_{j,\alpha'}\Big(\sum_{l=1}^{n}\sum_{k,\alpha}(S_{j,\alpha';k,\alpha}\big(p_k^{(\alpha)}(\lambda_l) - c_k^{(\alpha)}\big)\Big)^2 + O(e^{-d\log^2 n}),\end{aligned}$$

where

$$c_k^{(\alpha)} = \frac{n}{n_\alpha}(p_k^{(\alpha)}, \rho^{(\alpha)}).$$

Using the representations

$$e^{\beta x^2/2} = \sqrt{\frac{\beta}{2\pi}} \int du e^{\beta xu/2 - \beta u^2/8}, \quad e^{-\beta x^2/2} = \sqrt{\frac{\beta}{2\pi}} \int du e^{i\beta xu/2 - \beta u^2/8}, \tag{3.16}$$

we obtain

$$Q_{\bar{n}}[h] := \Big(\frac{\beta}{2\pi}\Big)^{Mq} \mathcal{Z}_{n,\beta}[V] e^{n^2\beta\Sigma^*/2} \int \prod_{\alpha=1}^{q}\prod_{k=1}^{M} du_{k,\alpha}^{(1)} du_{k,\alpha}^{(2)} e^{-\frac{\beta}{8}(\bar{u},\bar{u})} I_{\bar{n}}(\bar{u}), \tag{3.17}$$

where $\bar{u} := (\bar{u}^{(1)}, \bar{u}^{(2)})$,

$$\begin{aligned} I_{\bar{n}}(\bar{u}) :=& \mathcal{Z}_{n,\beta}^{-1}[V] e^{-n^2\beta\Sigma^*/2} \prod \frac{Q_{n_\alpha}[\mu_\alpha^{-1} V_\alpha^{(a)} - n_\alpha^{-1}\widetilde{h}_\alpha]}{n_\alpha!}, \\ \widetilde{h}_\alpha(\lambda) =& (n_\alpha/\mu_\alpha - n) V_\alpha^{(a)} + h_\alpha(\lambda) + s^{(\alpha)}(\bar{u},\lambda) - \frac{n}{n_\alpha}\big(s^{(\alpha)}(\bar{u},.), \rho_\alpha), \\ s^{(\alpha)}(\bar{u},\lambda) :=& \sum_{j,k,\alpha'} \Big(\widehat{S}_{j,\alpha';k,\alpha} u_{j,\alpha'}^{(1)} + iS_{j,\alpha';k,\alpha} u_{j,\alpha'}^{(2)}\Big) p_k^{(\alpha)}(\lambda). \end{aligned} \tag{3.18}$$

We are going to apply (1.16) to $Q_{n_\alpha}[\mu_\alpha^{-1} V_\alpha^{(a)} - n_\alpha^{-1}\widetilde{h}_\alpha]$. According to Theorem 1, it can be done for those $\bar{u} := (\bar{u}^{(1)}, \bar{u}^{(2)})$ which provide the condition

$$U_1 = \{\bar{u} := (\bar{u}^{(1)}, \bar{u}^{(2)}) : \sum_\alpha |(D_\alpha \widetilde{h}_\alpha, \widetilde{h}_\alpha)| \le c_0 \log n\}. \tag{3.19}$$

Remark that evidently $||\widetilde{h}_\alpha^{(6)}||_\infty \le CM^7 = C\log^{14} n$. It will be proven below (see Lemma 3) that the integral over the compliment of U_1 gives us $O(n^{-\kappa})$, so we should concentrate on $\bar{u} \in U_1$.

For $\bar{u} \in U_1$ (1.16) implies

$$\begin{aligned} \frac{Q_{n_\alpha}[\mu_\alpha^{-1} V_\alpha^{(a)} - n_\alpha^{-1}\widetilde{h}_\alpha]}{n_\alpha!} =& \exp\Big\{\frac{\beta}{2}(\frac{n_\alpha}{\mu_\alpha})^2 \mathcal{E}_\alpha + F_\beta(n_\alpha) + n_\alpha(\frac{\beta}{2}-1)\big((\log\frac{\rho_\alpha}{\mu_\alpha}, \frac{\rho_\alpha}{\mu_\alpha}) - 1 - \log 2\pi\big) \\ &+ \frac{\beta}{2} r_\beta[\mu_\alpha^{-1}\rho_\alpha] + \frac{\beta}{2}\Big(\widetilde{h}_\alpha, \frac{n_\alpha}{\mu_\alpha}\rho_\alpha + (\frac{2}{\beta}-1)\nu_\alpha\Big) + \frac{\beta}{8}\Big(D_\alpha\widetilde{h}_\alpha, \widetilde{h}_\alpha\Big)\Big\}, \end{aligned}$$

where $r_\beta[\rho]$ is defined in (1.20), $F_\beta(n)$ is defined in (1.21), and $\mathcal{E}_\alpha$ is the energy, corresponding to the potential $V_\alpha^{(a)}$ on σ_α. In view of (3.4) we have

$$\mathcal{E}_\alpha = L[\rho_\alpha, \rho_\alpha] - (V_\alpha^{(a)}, \rho_\alpha) = -L[\rho_\alpha, \rho_\alpha].$$

The definition of $\widetilde{h}_\alpha$ (3.18) and (3.4) yield

$$\begin{aligned} \big(\widetilde{h}_\alpha, \frac{n_\alpha}{\mu_\alpha}\rho_\alpha + (\frac{2}{\beta}-1)\nu_\alpha\big) =& \big(h_\alpha, \frac{n_\alpha}{\mu_\alpha}\rho_\alpha + (\frac{2}{\beta}-1)\nu_\alpha\big) + 2\frac{n_\alpha}{\mu_\alpha}\Big(\frac{n_\alpha}{\mu_\alpha} - n\Big) L[\rho_\alpha, \rho_\alpha] \\ &+ \Big(\frac{n_\alpha}{\mu_\alpha} - n\Big)(\frac{2}{\beta}-1)(V_\alpha^{(a)}, \nu_\alpha) + \big(s^{(\alpha)}(u), (\frac{n_\alpha}{\mu_\alpha} - n\big)\rho_\alpha + (\frac{2}{\beta}-1)\nu_\alpha\big). \end{aligned}$$

Moreover, (1.22) implies that

$$DL\rho_\alpha = -\rho_\alpha + \frac{\mu_\alpha}{\pi} X_\alpha^{-1/2},$$

where here and below we denote $X_\alpha^{-1/2} = |X_{\sigma_\alpha}^{-1/2}|\mathbf{1}_{\sigma_\alpha}$ with X_σ of (1.14). Hence, using (3.4), we obtain

$$\begin{aligned}
\frac{1}{4}\Big(\bar D_\alpha \widetilde h_\alpha, \widetilde h_\alpha\Big) =& \frac{1}{4}\Big(\bar D_\alpha h_\alpha, h_\alpha\Big) + \Big(\frac{n_\alpha}{\mu_\alpha} - n\Big)\Big(\frac{\mu_\alpha}{\pi} X_\alpha^{-1/2} - \rho_\alpha, h_\alpha\Big) \\
&+ \Big(\frac{n_\alpha}{\mu_\alpha} - n\Big)^2\Big(- L[\rho_\alpha, \rho_\alpha] + \frac{\mu_\alpha^2}{\pi^2} L[X_\alpha^{-1/2}, X_\alpha^{-1/2}]\Big) \\
&\Big(s^{(\alpha)}(u), \Big(\frac{n_\alpha}{\mu_\alpha} - n\Big)\Big(\frac{\mu_\alpha}{\pi} X_\alpha^{-1/2} - \rho_\alpha\Big) + \frac{Dh}{2}\Big) + \frac{1}{4}\Big(\bar D_\alpha s^{(\alpha)}(u), s^{(\alpha)}(u)\Big).
\end{aligned}$$

In addition,

$$-\big(\nu_\alpha, \mu_\alpha^{-1} V_\alpha^{(a)}\big) + \big(\log\frac{\rho_\alpha}{\mu_\alpha}, \frac{\rho_\alpha}{\mu_\alpha}\big) = \frac{1}{\pi}(\log\frac{\rho_\alpha}{\mu_\alpha}, X_\alpha^{-1/2}).$$

Hence, if we introduce the notations

$$\begin{aligned}
&X_{\bar n}^{-1/2} = \pi^{-1}\big((n_1 - n\mu_1)X_1^{-1/2}, \dots, (n_q - n\mu_q)X_q^{-1/2}\big), \\
&s(u) = (s^{(1)}(u), \dots, s^{(q)}(u)), \\
&h = (h_1, \dots, h_q), \quad \nu = (\nu_1, \dots, \nu_q), \quad T = \big(\log\frac{\rho_1}{\mu_1}, \dots, \log\frac{\rho_q}{\mu_q}\big),
\end{aligned} \tag{3.20}$$

then for $\bar u \in U_1$ we obtain finally

$$\begin{aligned}
I_{\bar n}(\bar u) =& I_{\bar n}^{(0)} \cdot I_{\bar n}^{(1)} \cdot I_{\bar n}^{(2)}(u)(1 + O(n^{-\kappa})), \quad \bar u \in U_1, && (3.21)\\
I_{\bar n}^{(0)} =& \exp\Big\{ - n\big(\frac{\beta}{2} - 1\big)\sum \mu_\alpha \log\mu_\alpha + \sum F_\beta(n_\alpha) - F_\beta(n)\Big\}, \\
I_{\bar n}^{(1)} =& \exp\Big\{\frac{\beta}{8}\big(\bar D h, h\big) + \frac{\beta}{2}\big(\widehat L X_{\bar n}^{-1/2}, X_{\bar n}^{-1/2})\Big) + \big(\frac{\beta}{2} - 1\big)\big(T, X_{\bar n}^{-1/2}\big) \\
&+ \big(h, \frac{\beta}{2} X_{\bar n}^{-1/2} - \big(\frac{\beta}{2} - 1\big)\nu\big)\Big\}, \\
I_{\bar n}^{(2)}(\bar u) =& \exp\Big\{\frac{\beta}{8}(\bar D s(u), s(u)) + \frac{\beta}{2}\Big(s(u), X_{\bar n}^{-1/2} + \frac{1}{2}\bar D h + (\frac{2}{\beta} - 1)\nu\Big)\Big\}.
\end{aligned}$$

To integrate with respect to $\bar u$, we introduce the block matrices

$$\begin{aligned}
&E = \begin{pmatrix} I & I \\ I & I \end{pmatrix}, \quad \mathcal{D} = \bar D^{(M)} E = \begin{pmatrix} \bar D^{(M)} & \bar D^{(M)} \\ \bar D^{(M)} & \bar D^{(M)} \end{pmatrix} \quad \mathcal{S} = \begin{pmatrix} \widehat S & 0 \\ 0 & iS \end{pmatrix} \\
&\bar D^{(M)}_{\alpha,k;\alpha',k'} := \delta_{\alpha,\alpha'}\big(\bar D_\alpha p_k^{(\alpha)}, p_{k'}^{(\alpha)}\big), \quad \mathcal{F} = I - \mathcal{S}\mathcal{D}\mathcal{S}.
\end{aligned}$$

Thus,

$$e^{-\frac{\beta}{8}(\overline{u}, \overline{u})} I_{\bar n}^{(2)}(u) = \exp\Big\{ - \frac{\beta}{8}\Big(\mathcal{F}\overline{u}, \overline{u}\Big) + \frac{\beta}{4}\big(\mathcal{S}\bar u, \bar R^{(M)}\big)\Big\}, \tag{3.22}$$

where

$$\bar R^{(M)} := (\bar r^{(M)}, \bar r^{(M)}) \quad \bar r^{(M)} = \{r_{\alpha,k}^{(M)}\}, \quad r_{\alpha,k}^{(M)} := (2X_{\bar n}^{-1/2} + 2(1 - \frac{2}{\beta})\nu + \bar D h, p_k^{(\alpha)}). \tag{3.23}$$

Lemma 3 *There exist $\delta_1, \kappa_1 > 0$, such that*

$$\Re(\mathcal{F}\bar{u}, \bar{u}) \geq \delta_1(\bar{u}, \bar{u}), \tag{3.24}$$

and $I_{\bar{n}}$, $I_{\bar{n}}^{(1)}$, $I_{\bar{n}}^{(2)}$ defined by (3.21) satisfy the bounds

$$\begin{aligned}
&\Big(\frac{\beta}{2\pi}\Big)^{Mq} \int_{U_1^c} e^{-\beta(\bar{u},\bar{u})/8} |I_{\bar{n}}(\bar{u})| d\bar{u} \leq n^{-\kappa_1}, \\
&\Big(\frac{\beta}{2\pi}\Big)^{Mq} \int_{U_1^c} e^{-\beta(\bar{u},\bar{u})/8} |I_{\bar{n}}^{(1)} I_{\bar{n}}^{(2)}(\bar{u})| d\bar{u} \leq n^{-\kappa_1},
\end{aligned} \tag{3.25}$$

where U_1^c is a complement of U_1 from (3.19).

The lemma and (3.22) imply that the integral over $\bar{u}$ of $I_{\bar{n}}(u)$ coincides with the integral over $\bar{u}$ of $I_{\bar{n}}^{(1)} I_{\bar{n}}^{(2)}(\bar{u})$ up to the error $O(n^{-\kappa_1})$. By the virtue of the standard Gaussian integration formulas we obtain

$$\begin{aligned}
I_{\bar{n}}^* :=& \Big(\frac{\beta}{2\pi}\Big)^{Mq} \int e^{-\beta(\bar{u},\bar{u})/8} I_2(\bar{u}) du \\
=& \det{}^{-1/2}\mathcal{F} \exp\Big\{\frac{\beta}{8}\big(\mathrm{Tr}(\mathcal{S}\mathcal{F}^{-1}\mathcal{S}E)\bar{r}^{(M)}, \bar{r}^{(M)}\big)\Big\}(1 + O(n^{-\kappa})),
\end{aligned} \tag{3.26}$$

where $\bar{r}^{(M)}$ is defined in (3.23) and

$$\mathrm{Tr}(\mathcal{S}\mathcal{F}^{-1}\mathcal{S}E) = (\mathcal{S}\mathcal{F}^{-1}\mathcal{S}E)_{11} + (\mathcal{S}\mathcal{F}^{-1}\mathcal{S}E)_{22}.$$

But since for any A, B $\det(1 + AB) = \det(1 + BA)$, we obtain

$$\det \mathcal{F} = \det(I - \mathcal{S}\mathcal{D}\mathcal{S}) = \det \begin{pmatrix} I - \bar{D}^{(M)}\widehat{\mathcal{A}}^{(M)} & \bar{D}^{(M)}\mathcal{A}^{(M)} \\ -\bar{D}^{(M)}\widehat{\mathcal{A}}^{(M)} & I + \bar{D}^{(M)}\mathcal{A}^{(M)} \end{pmatrix} =: \det \mathcal{F}_1,$$

$$\begin{aligned}
\det \mathcal{F}_1 = \det(I + \bar{D}^{(M)}\mathcal{A}^{(M)}) \det\Big(1 - \bar{D}^{(M)}\widehat{\mathcal{A}}^{(M)} + \bar{D}^{(M)}\widehat{\mathcal{A}}^{(M)}(I + \bar{D}^{(M)}\mathcal{A}^{(M)})^{-1}\bar{D}^{(M)}\mathcal{A}^{(M)}\Big) \\
= \det(1 - \bar{D}(\widehat{\mathcal{A}}^{(M)} - \mathcal{A}^{(M)})) = \det(1 - \bar{D}\widetilde{\mathcal{L}}^{(M)}).
\end{aligned}$$

Similarly, since $\mathcal{S}\mathcal{F}^{-1}\mathcal{S} = S^2\mathcal{F}_1^{-1}$ and

$$\begin{pmatrix} I - \bar{D}\widehat{\mathcal{A}}^{(M)} & \bar{D}^{(M)}\mathcal{A}^{(M)} \\ -\bar{D}^{(M)}\widehat{\mathcal{A}}^{(M)} & I + \bar{D}^{(M)}\mathcal{A}^{(M)} \end{pmatrix}^{-1} = \mathcal{G}^{(M)} \begin{pmatrix} I + \bar{D}^{(M)}\mathcal{A}^{(M)} & -\bar{D}^{(M)}\mathcal{A}^{(M)} \\ \bar{D}^{(M)}\widehat{\mathcal{A}}^{(M)} & I - \bar{D}^{(M)}\widehat{\mathcal{A}}^{(M)} \end{pmatrix},$$

where $\mathcal{G}^{(M)} := \Big(1 - \bar{D}\widetilde{\mathcal{L}}^{(M)}\Big)^{-1}$, we have

$$\mathrm{Tr}(\mathcal{S}\mathcal{F}^{-1}\mathcal{S}E) = \mathrm{Tr}(\mathcal{S}^2\mathcal{F}_1^{-1}E) = \mathrm{Tr}(\mathcal{S}^2E)\mathcal{G}^{(M)} = \widetilde{\mathcal{L}}^{(M)}\mathcal{G}^{(M)}.$$

Hence we obtain for $I_{\bar{n}}^*$ of (3.26)

$$I_{\bar{n}}^* = \det{}^{1/2}\mathcal{G}^{(M)} \exp\Big\{\frac{\beta}{8}\Big(\widetilde{\mathcal{L}}^{(M)}\mathcal{G}^{(M)}\bar{r}^{(M)}, \bar{r}^{(M)}\Big)\Big\}(1 + O(n^{-\kappa})).$$

Using Proposition 1 and Lemma 3, we can now replace $\widetilde{\mathcal{L}}^{(M)}$ by the "block" integral operator $\widetilde{L}$ with zero diagonal blocks and off-diagonal blocks $\mathcal{L}_{\alpha,\alpha'} : L_2[\sigma_{\alpha',2\varepsilon}] \to L_2[\sigma_{\alpha,2\varepsilon}]$

$$\widetilde{\mathcal{L}}_{\alpha,\alpha'}[f] = \big(\mathbf{1}_{\sigma_{\alpha,2\varepsilon}}\widetilde{\mathcal{L}}\mathbf{1}_{\sigma_{\alpha',2\varepsilon}}\big)[f]$$

The error of this replacement is $O(e^{-c\log^2 n})$. Hence,

$$\begin{aligned} I^*_{\bar n} =&\det{}^{1/2}\mathcal{G}\exp\Big\{\frac{\beta}{8}\big(\mathcal{G}\bar D h,h\big)-\frac{\beta}{8}\big(\bar D h,h\big)+\frac{\beta}{2}\Big(\mathcal{G}\big(X_{\bar n}^{-1/2}+(\frac{2}{\beta}-1)\nu\big),\bar D h\Big) \\ &+\frac{\beta}{2}\Big(\widetilde{\mathcal{L}}\mathcal{G}\big(X_{\bar n}^{-1/2}+(\frac{2}{\beta}-1)\nu\big),\big(X_{\bar n}^{-1/2}+(\frac{2}{\beta}-1)\nu\big)\Big)\Big\}(1+O(n^{-\kappa})). \end{aligned} \tag{3.27}$$

Moreover, since the operator $\bar D$ is defined on σ (see (1.26) and (1.17)) and $X_{\bar n}^{-1/2}$ are also defined on σ, one can see that the operator $\widetilde{\mathcal{L}}$ appears in (3.27) in the combination $\mathbf{1}_\sigma\widetilde{\mathcal{L}}\mathbf{1}_\sigma$, so starting from this moment we assume that $\widetilde{\mathcal{L}}:\mathcal{H}\to\mathcal{H}$. Let us study

$$\psi_{\bar n}:=\mathcal{G}X_{\bar n}^{-1/2}\Rightarrow X_{\bar n}^{-1/2}=(1-\bar D\widetilde{\mathcal{L}})\psi_{\bar n}.$$

In view of (1.22) we get

$$\widehat{\mathcal{L}}X_{\bar n}^{-1/2}=\widehat{\mathcal{L}}(1-\bar D\widetilde{\mathcal{L}})\psi_{\bar n}=\widehat{\mathcal{L}}\psi_{\bar n}+\widetilde{\mathcal{L}}\psi_{\bar n}-(\widetilde{\mathcal{L}}\psi_{\bar n},X^{-1/2})\mathbf{1}_{\sigma_\alpha}=\mathcal{L}\psi_{\bar n}+\text{const}.$$

Thus we conclude that

$$(\mathcal{L}\psi_{\bar n})_\alpha(\lambda)=c_\alpha(\bar n)=\text{const}\Rightarrow\psi_{\bar n}=\sum c_\alpha(\bar n)\psi^{(\alpha)},$$

where $\psi^{(\alpha)}$ are defined in (1.29). Moreover, by (1.28)-(1.29) we have

$$\begin{aligned} &\sum_{\alpha'}\mathcal{Q}_{\alpha\alpha'}c_{\alpha'}(\bar n)=(\psi_{\bar n},\mathbf{1}_{\sigma_\alpha})=(\mathcal{G}X_{\bar n}^{-1/2},\mathbf{1}_{\sigma_\alpha}) \\ &=(X_{\bar n}^{-1/2},\mathbf{1}_{\sigma_\alpha})-(\widetilde{\mathcal{L}}\mathcal{G}X_{\bar n}^{-1/2},\bar D\mathbf{1}_{\sigma_\alpha})=(X_{\bar n}^{-1/2},\mathbf{1}_{\sigma_\alpha})=\Delta n_\alpha \\ &\Rightarrow\quad\psi_{\bar n}=\sum_{\alpha,\alpha'}\mathcal{Q}^{-1}_{\alpha\alpha'}\psi^{(\alpha')}\Delta n_\alpha. \end{aligned}$$

Now let us transform the last two terms S_3 and S_4 in the r.h.s. of (3.27).

$$\begin{aligned} S_3=&\frac{\beta}{2}\big(X_{\bar n}^{-1/2}+(\frac{2}{\beta}-1)\nu,\mathcal{G}^*(\widetilde{\mathcal{L}}\bar D-1+1)h\big)=\frac{\beta}{2}\big(X_{\bar n}^{-1/2}+(\frac{2}{\beta}-1)\nu,\mathcal{G}^*h-h\big) \\ =&-\frac{\beta}{2}(X_{\bar n}^{-1/2},h)+\frac{\beta}{2}(\psi_{\bar n},h)-(\frac{\beta}{2}-1)\big((\mathcal{G}-1)\nu,h\big), \\ S_4=&\frac{\beta}{2}(\widetilde{\mathcal{L}}\mathcal{G}X_{\bar n}^{-1/2},X_{\bar n}^{-1/2})+\frac{\beta}{2}\Big(\frac{2}{\beta}-1\Big)^2\big(\widetilde{\mathcal{L}}\mathcal{G}\nu,\nu\big)+2(\frac{\beta}{2}-1)\big(\widehat{\mathcal{L}}\psi_{\bar n},\nu\big), \end{aligned} \tag{3.28}$$

since $\big(\widehat{\mathcal{L}}\psi_{\bar n},\nu\big)=-\big(\widetilde{\mathcal{L}}\psi_{\bar n},\nu\big)$ in view of $(\mathcal{L}\psi_{\bar n})_\alpha=\text{const}$ and $(\nu_\alpha,\mathbf{1}_{\sigma_\alpha})=0$. Since by (1.22) $\widehat{\mathcal{L}}D=\widehat{\mathcal{L}}\bar D$, we obtain

$$\begin{aligned} 2\big(\widehat{\mathcal{L}}\psi_{\bar n},\nu_\alpha\big)=&\big(\mathbf{1}_{\sigma_\alpha}\log X_\alpha^{1/2},\psi_{\bar n}\big)-\log(d_\alpha/2)\big(\mathbf{1}_{\sigma_\alpha},\psi_{\bar n}\big)-\big(\widehat{\mathcal{L}}\bar D\log P_\alpha,\psi_{\bar n}\big) \\ =&\big(\mathbf{1}_{\sigma_\alpha}\log X_\alpha^{1/2},\psi_{\bar n}\big)-\log(d_\alpha/2)\big(\mathbf{1}_{\sigma_\alpha},\psi_{\bar n}\big)+\big(\log P_\alpha,\psi_{\bar n}\big)-\big(X_{\bar n}^{-1/2},\log P_\alpha\big) \\ =&\big(\mathbf{1}_{\sigma_\alpha}\log\frac{\rho_\alpha}{\mu_\alpha},\psi_{\bar n}\big)-\big(\mathbf{1}_{\sigma_\alpha}\log\frac{\rho_\alpha}{\mu_\alpha},X_{\bar n}\big). \end{aligned}$$

Thus

$$S_4 = \frac{\beta}{2}(\widetilde{\mathcal{L}}\mathcal{G}X_{\bar{n}}^{-1/2}, X_{\bar{n}}^{-1/2}) + \frac{\beta}{2}\Big(\frac{2}{\beta}-1\Big)^2(\widetilde{\mathcal{L}}\mathcal{G}\nu,\nu) + \big(\frac{\beta}{2}-1\big)\big(T, \psi_{\bar{n}} - X_{\bar{n}}^{-1/2}\big). \tag{3.29}$$

In addition, using that $\bar{D}\mathcal{L}\mathcal{G}X_{\bar{n}}^{-1/2}=0$, $\bar{D}\widehat{\mathcal{L}}X_{\bar{n}}^{-1/2}=0$, we have

$$\begin{aligned}(\mathcal{L}\psi_{\bar{n}},\psi_{\bar{n}}) =&(\mathcal{L}\mathcal{G}X_{\bar{n}}^{-1/2},\mathcal{G}X_{\bar{n}}^{-1/2}) = (\mathcal{L}\mathcal{G}X_{\bar{n}}^{-1/2},(1+\bar{D}\widetilde{\mathcal{L}}\mathcal{G})X_{\bar{n}}^{-1/2})\\ =&(\mathcal{L}\mathcal{G}X_{\bar{n}}^{-1/2},X_{\bar{n}}^{-1/2}) = ((1+\bar{D}\widetilde{\mathcal{L}}\mathcal{G})X_{\bar{n}}^{-1/2},\widehat{\mathcal{L}}X_{\bar{n}}^{-1/2}) + (\widetilde{\mathcal{L}}\mathcal{G}X_{\bar{n}}^{-1/2},X_{\bar{n}}^{-1/2})\\ =&(\widehat{\mathcal{L}}X_{\bar{n}}^{-1/2},X_{\bar{n}}^{-1/2}) + (\widetilde{\mathcal{L}}\mathcal{G}X_{\bar{n}}^{-1/2},X_{\bar{n}}^{-1/2}).\end{aligned}$$

This relation combined with (3.27), (3.28), (3.29), and (3.21) yields

$$\begin{aligned}I_{\bar{n}}^{(1)}I_{\bar{n}}^{*} =&\det{}^{1/2}\mathcal{G}\exp\Big\{\frac{2}{\beta}\big(\frac{\beta}{2}-1\big)^2(\widetilde{\mathcal{L}}\mathcal{G}\nu,\nu) + \frac{\beta}{8}(\mathcal{G}Dh,h) + \Big(\frac{\beta}{2}-1\Big)\Big(\mathcal{G}\nu,h\Big)\Big\}\\ &\cdot\exp\Big\{\frac{\beta}{2}\Big(\mathcal{L}\psi_{\bar{n}},\psi_{\bar{n}}\Big) + \frac{\beta}{2}(\psi_{\bar{n}},h) + \Big(\frac{\beta}{2}-1\Big)\big(\psi_{\bar{n}},T\big)\Big\}\big(1+O(n^{-\kappa})\big).\end{aligned}$$

Multiplying this by $I_{\bar{n}}^{(0)}$ from (3.18) and taking into account that

$$\begin{aligned}\sum F_\beta(n_\alpha) - F_\beta(n) =&\big(\frac{\beta}{2}-1\big)\sum\big(n\mu_\alpha\log\mu_\alpha + (n_\alpha - n\mu_\alpha)\log\mu_\alpha\big)\\ &- c_\beta\sum\log\mu_\alpha - c_\beta(q-1)\log n + O\big(\frac{||\Delta n||^2}{n}\big),\end{aligned}$$

$\sum n_\alpha = n$, $\sum\mu_\alpha = 1$, and n_α under consideration satisfy (3.11), we obtain (1.34).

4 Auxiliary results

Proof of Proposition 1. Assume that $k_\alpha \ge k_{\alpha'}$, and prove that

$$|I_k(\lambda)| := \Big|\mathbf{1}_{\sigma_{\alpha',2\varepsilon}}(\lambda)\int_{\sigma_{\alpha,2\varepsilon}}\log|\lambda-\mu|\frac{p_k^{(\alpha)}(\mu)}{|X_{\sigma_{\alpha,2\varepsilon}}^{1/2}(\mu)|}d\mu\Big| \le Ce^{-2dk}. \tag{4.1}$$

Then, using that

$$\int_{\sigma_{\alpha',2\varepsilon}}\frac{|p_k^{(\alpha)}(\lambda)|}{|X_{\sigma_{\alpha,2\varepsilon}}^{1/2}(\lambda)|}d\lambda \le 1,$$

we obtain (3.12), since $k+k' \le 2\max\{k,k'\}$. Changing the variables in the integral in (4.1) $\mu = c_\alpha + d_\alpha\cos x$ with $c_\alpha = \frac{1}{2}\big(a_\alpha+b_\alpha\big)$, $d_\alpha = \frac{1}{2}\big(b_\alpha - a_\alpha + 4\varepsilon\big)$, and integrating by parts, we obtain

$$\begin{aligned}I_k(\lambda) =&d_\alpha k^{-1}\int_0^\pi\frac{\sin x\sin kx}{z-\cos x}dx = d_\alpha(2k)^{-1}\int_0^\pi\frac{\cos(k-1)x-\cos(k+1)x}{z-\cos x}dx\\ =&\frac{d_\alpha}{8k\pi i}\oint\frac{\zeta^{k-1}-\zeta^{k+1}+\zeta^{-k+1}-\zeta^{-k-1}}{\zeta^2+1-2z\zeta}d\zeta\\ =&\frac{d_\alpha}{8k\pi i}\oint\frac{\zeta^{k-1}-\zeta^{k+1}}{\zeta^2+1-2z\zeta}d\zeta = d_\alpha\frac{\zeta^{k-1}(z)-\zeta^{k+1}(z)}{4k\sqrt{z^2-1}},\end{aligned}$$

where $z = (\lambda - c_\alpha)d_\alpha^{-1}$, $|z| > 1 + \delta_1$, $\zeta(z) = z - \sqrt{z^2 - 1}$, $|\zeta(z)| \le e^{-2d}$. This proves (4.1).$\square$

Proof of Lemma 2. Consider the Fourier transform $\widehat{a}(k)$ of $a(|\lambda|)$. Integrating by parts, it is easy to get that

$$
\begin{aligned}
k\widehat{a}(k) = \int_0^\infty a(\lambda)\sin k\lambda d\lambda = \frac{16}{(kd)^3} - \frac{24\sin kd}{(kd)^4} + 24\int_{kd}^\infty \frac{\sin t}{t^5}dt \qquad (4.2)\\
= 24\int_{kd}^\infty \frac{2t + t\cos t - 3\sin t}{t^5}dt.
\end{aligned}
$$

Here the last equality can be obtained by the differentiation of the both parts with respect to kd. Let us check that the numerator in the last integral is positive. Indeed, it is 0 at $t = 0$, its derivative is positive on $(0, \pi)$, and it is evidently positive for $t \ge \pi$. Hence we get the first assertion of the lemma. To prove the second assertion, let us note that, applying the Taylor formula to the function $a_1(\lambda) := \lambda^{-1} - a'(\lambda)$ at the point $\lambda_0 = d$, we get $a_1(\lambda) = (\lambda - d)^4\xi^{-5}(\lambda) > 0$, and $a_1'(\lambda) < 0$ for $0 < \lambda < d$. Hence the Fourier transform of $l(|\lambda|) - a(|\lambda|)$ is

$$
\begin{aligned}
\widehat{l}(|\lambda|) - \widehat{a}(|\lambda|) =& \frac{1}{k}\int_0^\infty a_1(\lambda)\sin k\lambda d\lambda\\
=& \frac{1}{k^2}\sum_{j=0}^\infty \int_0^\pi \Big(a_1((t + 2j\pi)/k) - a_1((t + (2j + 1\pi))/k\Big)\sin t dt > 0.
\end{aligned}
$$

$\square$

Proof of Lemma 3 It is easy to see that, to prove (3.24), it suffices to show that

$$
\widehat{\mathcal{S}}_{\alpha\alpha}\overline{D}_\alpha^{(M)}\widehat{\mathcal{S}}_{\alpha\alpha} \le (1 - \delta_1) \Leftrightarrow \widehat{\mathcal{A}}_{\alpha\alpha}^{(M)}\overline{D}_\alpha^{(M)}\widehat{\mathcal{A}}_{\alpha\alpha}^{(M)} \le (1 - \delta_1)\widehat{\mathcal{A}}_{\alpha\alpha}^{(M)}. \qquad (4.3)
$$

Fix some α and denote $A := \widehat{\mathcal{A}}_{\alpha\alpha}$, $D := \overline{D}_\alpha$ and $L := \widehat{\mathcal{L}}_\alpha$ the complete matrices, corresponding to the above operators. Write them as a block matrices

$$
A = \begin{pmatrix} A^{(11)} & A^{(12)} \\ A^{(21)} & A^{(22)} \end{pmatrix}, \; D = \begin{pmatrix} D^{(11)} & D^{(12)} \\ D^{(21)} & D^{(22)} \end{pmatrix}, \; L = \begin{pmatrix} L^{(11)} & L^{(12)} \\ L^{(21)} & L^{(22)} \end{pmatrix},
$$

such that $A^{(11)} =: \widehat{\mathcal{A}}_{\alpha\alpha}^{(M)}$, $D^{(11)} = \overline{D}_\alpha^{(M)}$, and $L^{(11)} = \widehat{\mathcal{L}}_\alpha^{(M)}$. Below we will use the inequality valid for any block matrix $B \ge 0$

$$
B = \begin{pmatrix} B^{(11)} & B^{(12)} \\ B^{(21)} & B^{(22)} \end{pmatrix}, \quad B^{(21)}(B^{(11)})^{-1}B^{(12)} \le B^{(22)}. \qquad (4.4)
$$

Assume that we have proved the inequality

$$
D \le (1 - \delta_1)A^{-1} \quad \Leftrightarrow \quad ADA \le (1 - \delta_1)A \quad \Rightarrow \quad (ADA)^{(11)} \le (1 - \delta_1)A^{(11)}. \qquad (4.5)
$$

Then we get

$$
\begin{aligned}
(A^{(11)}D^{(11)}A^{(11)}f, f) =& ((ADA)^{(11)}f, f) - ((ADA)^{(22)}f, f) - 2\Re(A^{(12)}D^{(21)}A^{(11)}f, f)\\
\le& ((ADA)^{(11)}f, f) - 2\Re(A^{(12)}D^{(21)}A^{(11)}f, f). \qquad (4.6)
\end{aligned}
$$

But
$$|(A^{(12)}D^{(21)}A^{(11)}f,f)| \le ||(A^{(11)})^{-1/2}A^{(12)}D^{(21)}(A^{(11)})^{1/2}||\,(A^{(11)}f,f).$$
In addition, using (4.4) for the matrix A, we get
$$\begin{aligned}&||(A^{(11)})^{-1/2}A^{(12)}D^{(21)}(A^{(11)})^{1/2}||^2\\ =&||(A^{(11)})^{1/2}D^{(12)}A^{(21)}(A^{(11)})^{-1}A^{(12)}D^{(21)}(A^{(11)})^{1/2}||\\ \le&||(A^{(11)})^{1/2}D^{(12)}A^{(22)}D^{(21)}(A^{(11)})^{1/2}|| = ||(A^{(22)})^{1/2}D^{(21)}A^{(11)}D^{(12)}(A^{(22)})^{1/2}||.\end{aligned}$$
Then, taking into account that for any small enough $\varepsilon > 0$ (4.10) implies that $(-L^{(11)}) \le (D^{(11)}+\varepsilon)^{-1}$, we can use (4.4) for $D+\varepsilon$ in order to get
$$D^{(12)}A^{(11)}D^{(12)} \le D^{(21)}(-L^{(11)})D^{(12)} \le D^{(21)}(D^{(11)}+\varepsilon)^{-1}D^{(12)} \le D^{(22)}+\varepsilon.$$
Hence we obtain
$$\begin{aligned}&||(A^{(11)})^{-1/2}A^{(12)}D^{(21)}(A^{(11)})^{1/2}||^2 \le ||(A^{(22)})^{1/2}D^{(22)}(A^{(22)})^{1/2}||\\ \le&\mathrm{Tr}\,(A^{(22)})^{1/2}D^{(22)}(A^{(22)})^{1/2} = \mathrm{Tr}\,A^{(22)}D^{(22)}.\end{aligned} \tag{4.7}$$
Integrating by parts it is easy to check that
$$\begin{aligned}k^2j^2A_{k,j} =&k^2j^2\int_0^\pi\int_0^\pi a\Big(d_q(\cos x-\cos y)\Big)\cos kx\cos jy\,dxdy\\ =&-d_q^3a'''(0)\int_0^\pi\sin^3x\cos kx\cos jxdx + \int_0^\pi\int_0^\pi\tilde{a}(x,y)\cos kx\cos jy\,dxdy,\end{aligned} \tag{4.8}$$
where $d_q = \frac{1}{2}(b_q-a_q+4\varepsilon)$ and $\tilde{a}(x,y)$ is some bounded piece-wise continuous function. Hence we conclude that there exists a constant C_0 such that if we introduce the diagonal matrix A_d with the entries $(A_d)_{jk} = \delta_{jk}k^{-4}$, then
$$A_d^{-1/2}AA_d^{-1/2} \le C_0 \quad \Rightarrow \quad A \le C_0A_d.$$
Moreover, it is easy to check that there exists $C_1 > 0$ such that
$$D_{kk}^{(22)} \le C_1k^2. \tag{4.9}$$
Thus, from (4.7) and above bounds we obtain that
$$||(A^{(11)})^{-1/2}A^{(12)}D^{(21)}(A^{(11)})^{1/2}||^2 \le C_0\mathrm{Tr}\,A_d^{(22)}D^{(22)} = C\sum_{k=M+1}^{\infty}k^{-2} \le O(M^{-1}).$$
Finally we have from (4.6) and (4.5)
$$\begin{aligned}(A^{(11)}D^{(11)}A^{(11)}f,f) &\le ((ADA)^{(11)}f,f) + (A^{(11)}f,f)O(M^{-1/2})\\ &\le (1-\delta_1+O(M^{-1/2}))(A^{(11)}f,f) \le (1-\delta_1/2)(A^{(11)}f,f).\end{aligned}$$
Hence we need only to prove (4.5). Since the last relations of (1.22) yields
$$(\overline{D}_\alpha v,v) = ((-\widehat{\mathcal{L}}_\alpha)^{-1}v,v) + \pi^{-2}(v,X_\alpha^{-1/2})^2(\widehat{\mathcal{L}}_\alpha^{-1}\mathbf{1}_{\sigma_\alpha},\mathbf{1}_{\sigma_\alpha}) \le ((-\widehat{\mathcal{L}}_\alpha)^{-1}v,v), \tag{4.10}$$

it suffices to prove that

$$(-\widehat{\mathcal{L}})^{-1} \le (1-\delta_1)\widehat{\mathcal{A}}^{-1} \Leftrightarrow \widehat{\mathcal{A}} \le (1-\delta_1)(-\widehat{\mathcal{L}}). \tag{4.11}$$

But the last bound is a corollary of the following inequality for the Fourier transforms of $a(|\lambda|)$ and $\log|\lambda|^{-1}$

$$\widehat{a}(k) < (1-\delta_1)\widehat{l}(k) = (1-\delta_1)\pi/k.$$

Since we have already proved this inequality for $\delta_1 = 0$ in Lemma 2, we have $\widehat{a}(k)/\widehat{l}(k) < 1$. Besides, it follows from (4.2) that $\widehat{a}(k) \sim k^{-4}$, hence $\widehat{a}(k)/\widehat{l}(k) \to 0$, as $k \to \infty$, and moreover, $\widehat{a}(k)/\widehat{l}(k) \to 0$, as $k \to 0$. Thus there exists $\delta_1 > 0$ such that

$$\sup_{k>0} \widehat{a}(k)/\widehat{l}(k) = 1-\delta_1.$$

To prove the first relation of (3.25), we represent

$$\begin{aligned}
U_1^c \subset & U_2 \cup U_3 \cup U_4 \cup U_5, \qquad (4.12)\\
U_2 =& \{\bar u : \sum_\alpha (\widehat{\mathcal{S}}_\alpha D_{\alpha,\alpha}\widehat{\mathcal{S}}_\alpha u^{(1)}, u^{(1)}) \le \frac{c_0}{2}\log n \wedge (\mathcal{S}_\alpha D_{\alpha,\alpha}\mathcal{S}_\alpha u^{(2)}, u^{(2)}) \le \frac{c_0}{2}\log n\},\\
U_3 =& \{\bar u : \frac{c_0}{2}\log n \le \sum_\alpha (\widehat{\mathcal{S}}_\alpha D_{\alpha,\alpha}\widehat{\mathcal{S}}_\alpha u^{(1)}, u^{(1)}) \le n\log^2 n\},\\
U_4 =& \{\bar u : (u^{(1)}, u^{(1)}) \le C_* n^2 \wedge \sum_\alpha (\widehat{\mathcal{S}}_\alpha D_{\alpha,\alpha}\widehat{\mathcal{S}}_\alpha u^{(1)}, u^{(1)}) \ge n\log^2 n\},\\
U_5 =& \{\bar u : (u^{(1)}, u^{(1)}) \ge C_* n^2\}.
\end{aligned}$$

It is evident that

$$\begin{aligned}
Q_{n_\alpha}[\mu_\alpha^{-1}V_\alpha^{(a)} - n_\alpha^{-1}\widetilde{h}_\alpha] \le & |\sigma_\alpha|^{n_\alpha} \exp\Big\{\beta n_\alpha^2 \max\{\mu_\alpha^{-1}V_\alpha^{(a)} - n_\alpha^{-1}\Re\widetilde{h}_\alpha|\}/2\Big\}\\
\le & |\sigma_\alpha|^{n_\alpha} \exp\Big\{\beta(n_\alpha^2 C + n_\alpha \max\{|\Re\widetilde{h}_\alpha|\})/2\Big\}.
\end{aligned}$$

Moreover, the definition of $\widetilde{h}_\alpha$ (see (3.18)) and the Schwarz inequality yield

$$\begin{aligned}
|\Re\widetilde{h}_\alpha| \le & C_1\Big(1 + \max\Big|\sum_{j,k,\alpha'} \widehat{S}_{j,\alpha';k,\alpha} u^{(1)}_{j,\alpha'} p_j^{(\alpha)}(\lambda)\Big|\Big) \qquad (4.13)\\
\le & C_1\Big(1 + |u^{(1)}| \max \sum_j |p_j^{(\alpha)}(\lambda)| \sum_{k,\alpha'} |\widehat{S}_{j,\alpha';k,\alpha}|^2\Big)\\
\le & C_1 + C_2|u^{(1)}| \sum_j \widehat{\mathcal{A}}_{jj}^{1/2} \le C_1 + C_3|u^{(1)}|,
\end{aligned}$$

where the last inequality is based on the fact that $\widehat{\mathcal{A}}_{jj} \le Cj^{-4}$ in view of (4.8). Hence, choosing $C_* = \beta C_3$, we obtain

$$\Big(\frac{\beta}{2\pi}\Big)^{Mq} \int_{U_5} e^{-\beta(\bar u, \bar u)/8} I(u)du \le e^{-n^2 c}.$$

Similarly to (4.13), we have

$$\begin{aligned}
|\Re\widetilde{h}_\alpha(\lambda_1) - \Re\widetilde{h}_\alpha(\lambda_2)| \le & \sum_j |p_j^{(\alpha)}(\lambda_1) - p_j^{(\alpha)}(\lambda_2)| \widehat{\mathcal{A}}_{jj}^{1/2}\\
\le C|u^{(1)}||\lambda_1-\lambda_2|^{1/2} \sum_j j^{1/2}\widehat{\mathcal{A}}_{jj}^{1/2} \le & C'|u^{(1)}||\lambda_1-\lambda_2|^{1/2}.
\end{aligned}$$

Thus $n_\alpha^{-1}\widetilde{h}_\alpha(\lambda)$ is a Holder function for $\bar{u}\in U_4$, and we can use the result of [3], according to which

$$\begin{aligned} &Q_{n_\alpha}[\mu_\alpha^{-1}V_\alpha^{(a)}-n_\alpha^{-1}\Re\widetilde{h}_\alpha]\\ &\le \exp\Big\{\frac{\beta n_\alpha^2}{2}\max_{m\in\mathcal{M}_1^+[\sigma_{\alpha,\varepsilon}]}\{L[m,m]-(m,\mu_\alpha^{-1}V_\alpha^{(a)}-n_\alpha^{-1}\widetilde{h}_\alpha)\}+Cn\log n\Big\}, \end{aligned}$$

where $\mathcal{M}_1^+[\sigma_{\alpha,\varepsilon}]$ is a set of positive unit measures with supports belonging to $\sigma_{\alpha,\varepsilon}$. Since

$$-\mu_\alpha^{-1}V_\alpha^{(a)}(\lambda)\le -2\mu_\alpha^{-1}L[\rho_\alpha](\lambda),\quad \lambda\in\sigma_{\alpha,\varepsilon},$$

we have

$$\begin{aligned} &\max_{m\in\mathcal{M}_1^+[\sigma_{\alpha,\varepsilon}]}\{L[m,m]-(m,\mu_\alpha^{-1}V_\alpha^{(a)}-n_\alpha^{-1}\Re\widetilde{h}_\alpha)\}\\ \le&\max_{m\in\mathcal{M}_1^+[\sigma_{\alpha,\varepsilon}]}\{L[m,m]-(m,2\mu_\alpha^{-1}L[\rho_\alpha]-n_\alpha^{-1}\Re\widetilde{h}_\alpha)\}\\ \le&\max_{m\in\mathcal{M}_1[\sigma_{\alpha,\varepsilon}]}\{L[m,m]-(m,2\mu_\alpha^{-1}L[\rho_\alpha]-n_\alpha^{-1}\Re\widetilde{h}_\alpha)\}. \end{aligned}\tag{4.14}$$

Here $\mathcal{M}_1[\sigma_{\alpha,\varepsilon}]$ is a set of all signed unit measures with supports belonging to $\sigma_{\alpha,\varepsilon}$. It is easy to see that, if we remove the condition of positivity of measures, then the maximum point ρ_1 is uniquely defined by the conditions:

$$2L[\rho_1](\lambda)-2\mu_\alpha^{-1}L[\rho_\alpha](\lambda)-n_\alpha^{-1}\Re\widetilde{h}_\alpha(\lambda)=\text{const},\quad \lambda\in\sigma_{\alpha,\varepsilon},\quad \int_{\sigma_{\alpha,\varepsilon}}\rho_1=1.$$

Hence $\rho_1=\mu_\alpha^{-1}\rho_\alpha+\frac{1}{2}D_{\sigma_{\alpha,\varepsilon}}\widetilde{h}_\alpha$ and the r.h.s. of (4.14) takes the form

$$E_\alpha(\bar{u}):=-\mu_\alpha^{-2}L[\rho_\alpha,\rho_\alpha]+\frac{n_\alpha^{-2}}{4}(D_{\sigma_{\alpha,\varepsilon}}\Re\widetilde{h}_\alpha,\Re\widetilde{h}_\alpha)+n_\alpha^{-1}(\widetilde{h}_\alpha,\mu_\alpha^{-1}\rho_\alpha).$$

But by the definition of $\widetilde{h}_\alpha$ (see (3.18))

$$n_\alpha^{-1}(\widetilde{h}_\alpha,\mu_\alpha^{-1}\rho_\alpha)=O(n_\alpha^{-1})+O(n/n_\alpha-\mu_\alpha^{-1})=O(n^{-1}\log n).$$

Hence

$$E_\alpha(\bar{u})=-\mu_\alpha^{-2}L[\rho_\alpha,\rho_\alpha]+\frac{n_\alpha^{-2}}{4}(\widehat{\mathcal{S}}_\alpha D_{\sigma_{\alpha,\varepsilon}}\widehat{\mathcal{S}}_\alpha u^{(1)},u^{(1)})+O(n^{-1}\log n).$$

Thus,

$$I(u)\le\exp\{\frac{\beta}{8}(\widehat{\mathcal{S}}_\alpha D_{\sigma_{\alpha,\varepsilon}}\widehat{\mathcal{S}}_\alpha u^{(1)},u^{(1)})+O(n\log n)\},$$

and using the Chebyshev inequality and (4.3), we can conclude that for small enough n-independent $\tau>0$

$$\begin{aligned} &\Big(\frac{\beta}{2\pi}\Big)^{Mq}\int_{U_4}e^{-\beta(u,u)/8}I(u)du \qquad (4.15)\\ \le&\Big(\frac{\beta}{2\pi}\Big)^{Mq}\int e^{-\beta(u,u)/8}I(u)e^{\tau(\sum_\alpha(\widehat{\mathcal{S}}_\alpha D_{\alpha,\alpha}\widehat{\mathcal{S}}_\alpha u^{(1)},u^{(1)})-n\log^2 n)}\\ \le&\Big(\frac{\beta}{2\pi}\Big)^{Mq}\int e^{-\beta(u,u)/8}e^{\frac{\beta}{8}(\widehat{\mathcal{S}}_\alpha D_{\sigma_{\alpha,\varepsilon}}\widehat{\mathcal{S}}_\alpha u^{(1)},u^{(1)})+O(n\log n)}e^{\tau(\sum_\alpha(\widehat{\mathcal{S}}_\alpha D_{\alpha,\alpha}\widehat{\mathcal{S}}_\alpha u^{(1)},u^{(1)})-n\log^2 n)}\\ =&C(\tau)e^{-\tau n\log^2 n+cn\log n}. \end{aligned}$$

Note that (4.3) implies that the last integral is convergent, and since $\widehat{\mathcal{S}}_\alpha D_{\sigma_{\alpha,\varepsilon}}\widehat{\mathcal{S}}_\alpha$ is a trace class operator (see (4.8)–(4.9)), the additional quadratic form in the exponent changes the integral only by some uniformly bounded multiplier. For $u \in U_3$ (1.16) implies

$$\frac{Q_{n_\alpha}[\mu_\alpha^{-1}V_\alpha^{(a)} - n_\alpha^{-1}\Re\widetilde{h}_\alpha]}{Q_{n_\alpha}[\mu_\alpha^{-1}V_\alpha^{(a)}]} \le \exp\{(\widehat{\mathcal{S}}_\alpha D_{\alpha,\alpha}\widehat{\mathcal{S}}_\alpha u^{(1)}, u^{(1)}) + O(n^{-1}M(u^{(1)}, u^{(1)})^{3/2}) + C\}.$$

Hence, similarly to (4.15), for small enough n-independent $\tau > 0$ we have

$$\begin{aligned}&\Big(\frac{\beta}{2\pi}\Big)^{Mq}\int_{U_3} e^{-\beta(u,u)/8}I(u)du\\ \le&\Big(\frac{\beta}{2\pi}\Big)^{Mq}\int e^{-\beta(u,u)/8}I(u)e^{\tau(\sum_\alpha(\widehat{\mathcal{S}}_\alpha D_{\alpha,\alpha}\widehat{\mathcal{S}}_\alpha u^{(1)},u^{(1)})-c_0\log n)} = C(\tau)e^{-c_0\tau\log n/2}.\end{aligned}$$

By the same way we also obtain for U_2

$$\begin{aligned}&\Big(\frac{\beta}{2\pi}\Big)^{Mq}\int_{U_2} e^{-\beta(u,u)/8}I(u)du\\ \le&\Big(\frac{\beta}{2\pi}\Big)^{Mq}\int e^{-\beta(u,u)/8}I(u)e^{\tau(\sum_\alpha(\mathcal{S}_\alpha D_{\alpha,\alpha}\mathcal{S}_\alpha u^{(2)},u^{(2)})-c_0\log n)} = C(\tau)e^{-c_0\tau\log n/2}.\end{aligned}$$

This completes the proof of the first bound of (3.25). To prove the second bound we just use the Chebyshev inequality like above for U_2 and U_3. Lemma 3 is proved. □

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